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The Compactified Moduli Spaces of Pencils of Quadrics and Hyperelliptic Curves

Masterarbeit

Erlagung des akademischen Grades
Master of Science

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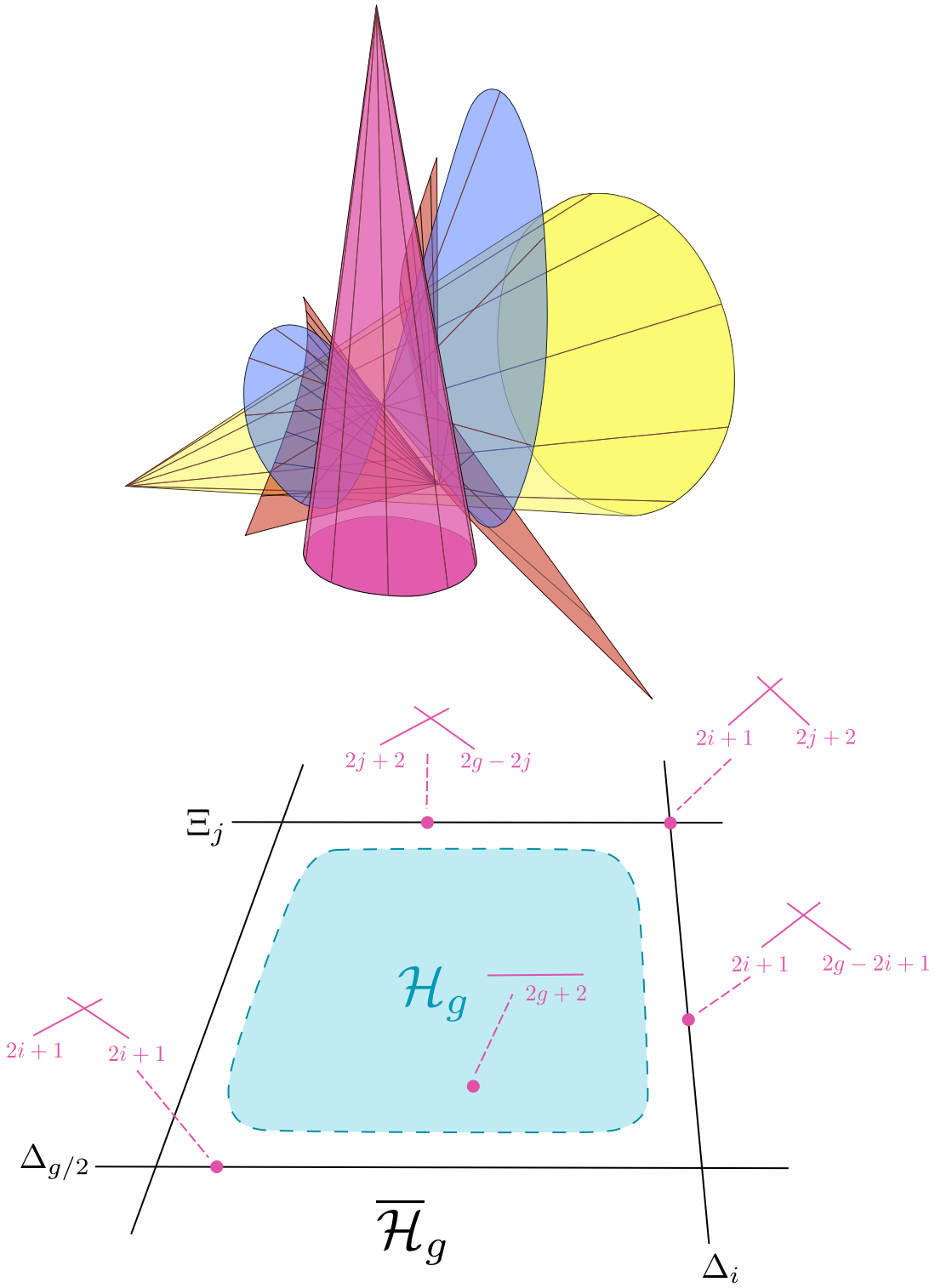


Figure 1: A pencil of quadrics and a cartoon of the moduli space of stable hyperelliptic curves with the corresponding stable marked curves shown.

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1 Introduction

A *pencil of quadrics in* $\mathbb{P}_n$ is defined by $\{\lambda A + \mu B \,|\, (\lambda : \mu) \in \mathbb{P}_1\}$, where A and B are symmetric $(n+1) \times (n+1)$ matrices. By abuse of notation, we write $P = \lambda A + \mu B$. A pencil of quadrics P is called *general* if its discriminant $\Delta P := \det(\lambda A + \mu B)$ admits no multiple root. Let $\mathcal{P}_n$ denote the (coarse) moduli space of general pencils of quadrics in $\mathbb{P}_n$, $\mathcal{B}_n$ denote the moduli space of binary forms of degree n with no multiple roots, and $\mathcal{H}_g$ denote the moduli space of smooth hyperelliptic curves. There is a canonical isomorphism between $\mathcal{P}_{2g+1}$ and $\mathcal{H}_g$ through $\mathcal{B}_{2g+2}$:

$$\mathcal{P}_{2g+1} \xrightarrow{\sim} \mathcal{B}_{2g+2} \xrightarrow{\sim} \mathcal{H}_g.$$

The map is constructed as follows: given a general pencil $P = \lambda A + \mu B$ in $\mathbb{P}^{2g+1}$, its discriminant ΔP is a binary form of degree $2g+2$. By viewing the binary form as a projective line with $2g+2$ distinct points, we obtain a smooth hyperelliptic curve by taking the double cover ramified at the $2g+2$ points. Such a hyperelliptic structure is unique (see Proposition 2.32), and by Riemann-Hurwitz (Theorem 2.31), is of genus g.

Recall that a vector bundle E of rank n and degree d is called *stable* (respectively *semistable*) if for every proper subbundle F of E, we have

$$\text{degree } F/\text{rank } F < (\text{resp. } \leq)\ d/n.$$

Let $U_L(n,d)$ denote the moduli space of stable vector bundles of rank n, degree d, with a fixed determinant L on X, an algebraic curve. It has been of classical interest to describe this moduli space. Notably, when X is a smooth hyperelliptic curve, Desale and Ramanan proved the following:

Theorem 1.1. [DR76] Let X be a smooth hyperelliptic curve with genus g. Let $\mathcal{P} = \lambda Q_1 + \mu Q_2$ be the nonsingular pencil of quadrics associated to it. Then for d odd, we have

$$U_L(2,d) \cong \{\mathbb{P}X \,|\, \dim(\mathbb{P}X) = g-2, \mathbb{P}X \subset Q_1 \cap Q_2\}.$$

In the same paper, Desale and Ramanan proved a similar result for the Jacobian of X.

Theorem 1.2. [DR76] Let X be a smooth hyperelliptic curve of genus g, $\mathcal{P} = \lambda Q_1 + \mu Q_2$ the nonsingular pencil of quadrics associated to it, and let $J(X)$ be the Jacobian of X. Then

$$J(C) \cong \{\mathbb{P}X \,|\, \dim(\mathbb{P}X) = g-1, \mathbb{P}X \subset Q_1 \cap Q_2\}.$$

Since then, extensions of these results have been made to certain nodal hyperelliptic curves. The case of rank 1, i.e., of the compactified Jacobian, was worked out in [Bho90]. [Bho98] and [Bho13] extended the result to the moduli space $U'_L(2,d)$ of semistable vector bundles of rank 2 with a fixed determinant L of degree d on a nodal hyperelliptic curve X of genus g. To extend this construction further to stable hyperelliptic curves, it is necessary to investigate first which curves correspond to which (semistable) pencils of quadrics.

The moduli spaces are constructed as a GIT quotient of algebraic varieties modulo the action of a reductive group: SL_{n+1} for $\mathcal{P}_n$, SL_2 for $\mathcal{B}_n$, and PGL_{6g-5} for $\mathcal{H}_g$. Hence,

they give rise to the notion of stability and semistability (see Section 2.2), allowing GIT compactifications $\overline{\mathcal{P}}_{2g+1}$, $\overline{\mathcal{B}}_{2g+2}$, and $\overline{\mathcal{H}}_g$. (Note that $\overline{\mathcal{H}}_g$ is usually defined as the closure of $\mathcal{H}_g$ in the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, so embedding the curve tricanonically leads to the action of PGL_{6g-5} on the corresponding Hilbert scheme.) These compactified moduli spaces are a priori not isomorphic. To answer the question of which pencils correspond to which curves is equivalent to finding a relation between $\overline{\mathcal{P}}_{2g+1}$ and $\overline{\mathcal{H}}_g$. Avritzer and Lange give the following answer:

Theorem I ([AL00, Thm 4.2])**.** The discriminant map induces an isomorphism

$$\overline{D} : \overline{\mathcal{P}}_n \xrightarrow{\sim} \overline{\mathcal{B}}_{n+1}.$$

However, the story is not as simple between $\overline{\mathcal{B}}_{2g+2}$ and $\overline{\mathcal{H}}_g$: $\overline{\mathcal{B}}_{2g+2}$ contains only one strictly semistable point (see Proposition 4.1), whereas $\overline{\mathcal{H}}_g \setminus \mathcal{H}_g = \Delta_1 \cup \cdots \cup \Delta_{[g/2]} \cup \Xi_0 \cup \cdots \cup \Xi_{[(g-1)/2]}$. A general member of Δ_i is represented by an admissible covering as shown in Figure 1, whereas a general element of Ξ_i is represented by an admissible covering as shown in Figure 2. We conclude that the geometry of $\overline{\mathcal{H}}_g$ does not allow an isomorphism to $\overline{\mathcal{B}}_{2g+2}$:

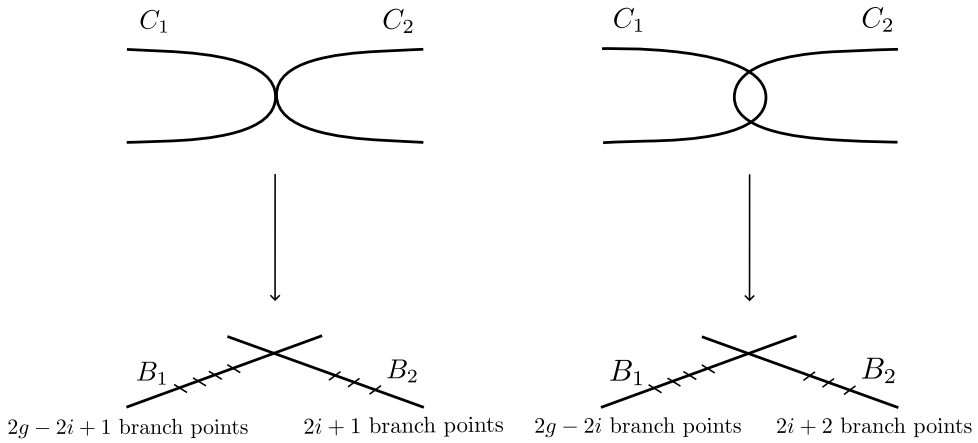


Figure 2: A general member of Δ_i on the left, and a general member of Ξ_i on the right.

Theorem II ([AL01, Cor 2.5 & Prop 5.1])**.** The canonical isomorphism $\mathcal{H}_g \xrightarrow{\sim} \mathcal{B}_{2g+2}$ extends to a holomorphic map

$$f_g : \overline{\mathcal{H}}_g \to \overline{\mathcal{B}}_{2g+2}.$$

In particular, f_g contracts Δ_i (respectively Ξ_i) to a subvariety of dimension $2g-2i-1$ (resp. $2g-2i-2$), except when $2i+1 = g+1$ (resp. $2i+2 = g+1$). In this case, f_g contracts Δ_i (resp. Ξ_i) to a point.

This thesis builds on the work of [AL00] and [AL01] to provide expositions for the two central theorems I and II. My goal is to provide an accessible visual guide to the theory. Towards this end, I included various drawings and examples to aid intuition. In addition to elaborating on the original arguments, my technical contributions include:

- corrected the typographical error in the statement of Theorem 4.3(2),
- simplified the proof of Theorem 4.3(1),
- revised the proof of Theorem 3.5(b), and

- revised the proof of Proposition 5.12.

The corrected statements and proofs involved only **minor typographical errors** that did not affect the validity of the results. The revised proofs were restructured for improved clarity and brevity, using alternative approaches where appropriate.

The content of this thesis is as follows: we recall the necessary background on moduli theory and, in particular, geometric invariant theory in Section 2. This can be skipped for readers familiar with the background. In Section 3, after reviewing the theory of pencils of quadrics, we determine the stable and semistable pencils. We prove Theorem I in Section 4 and Theorem II in Section 5.

2 Background

2.1 The Moduli Problem

In this section, we give a brief exposition on moduli theory, since we do not use the moduli structure of the spaces. For an introductory but comprehensive treatment, we refer the readers to the excellent lecture notes of [Sch20].

Denote by $\mathbf{Sch}_{\mathbb{C}}$ the category of schemes over $\mathbb{C}$, where for schemes $X, Y \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$ the set $\mathrm{Mor}(X, Y)$ of morphisms from X to Y is the set of morphisms $X \to Y$ over $\mathbb{C}$. Then for any scheme $M \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$ we have its *functor of points*

$$h^M : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}, \ X \mapsto \mathrm{Mor}(X, M).$$

Recall that the functor h^M uniquely determines the scheme M:

Lemma 2.1 (Yoneda's Lemma)**.** The functor

$$\begin{aligned} h^- : \mathbf{Sch}_{\mathbb{C}} &\to \mathrm{Functors}(\mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}) \\ M &\mapsto h^M \end{aligned}$$

is a fully faithful embedding.

By the Yoneda lemma, we can see the category of schemes as sitting inside the category of functors $\mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}$. Thus, we will give these functors their name.

Definition 2.2. A *moduli functor* h is a functor $h : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}$.

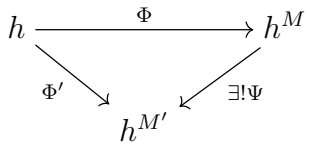
Definition 2.3. A moduli functor h is called *representable* if it is of the form $h \cong h^M$ for a scheme M. This scheme M (which is unique up to isomorphism) is called a *fine moduli space* for h.

Definition 2.4. Given a moduli functor h and a natural isomorphism $h \cong h^M$ for some scheme M, define the *universal family* $U \in h(M)$ to be the element in $h(M)$ corresponding to the canonical element $id_M \in \mathrm{Mor}(M, M) = h^M(M)$.

However, few functors are representable. One way to deal with this is to have a weaker notion for a scheme to "approximately represent" a functor.

Definition 2.5. Given a moduli functor h, a *coarse moduli space* is a pair (M, Φ) of a scheme M together with a natural transformation $\Phi : h \to h^M$ such that

(a) (M, Φ) is initial among all such pairs, i.e., for any other scheme M' and natural transformation $\Phi' : h \to h^{M'}$ there exists a unique natural transformation $\Psi : h^M \to h^{M'}$ such that the following diagram commutes



(b) The map Φ induces a bijection

$$\Phi(\mathrm{Spec}(\mathbb{C}) : h(\mathrm{Spec}(\mathbb{C})) \xrightarrow{\sim} h^M((\mathbb{C})) = M(\mathbb{C})$$

on $\mathbb{C}$-points.

2.2 Geometric Invariant Theory

In this section, we use geometric invariant theory to construct such quotient spaces. For more theory on the relations between GIT and moduli theory, we suggest the reader read the expositions by [Hos24] or [Laz13]. For more details on the construction or properties of the GIT quotients, we refer the reader to the foundational book by [MFK94].

2.2.1 Affine GIT Quotient

Let G be an affine algebraic group acting on an affine scheme $X = \mathrm{Spec}R$ by

$$g \cdot f(x) = f(g^{-1} \cdot x).$$

We wish to construct a "nice" quotient space X/G. Taking the naive topological quotient oftentimes results in a non-Hausdorff space. Ideally, we would like that

(a) The quotient X/G gives 1-to-1 parametrization of the G-orbits in X, and
(b) X/G has the structure of an affine scheme (such that $X \to X/G$ is a morphism that is constant on orbits).

The G action on X induces an action on the coordinate ring R. In the case that G is a reductive group, a foundational result of Hilbert says that the ring of invariants $R^G := \{r \in R : g \cdot r = r \ \forall g \in G\} \subset R$ is a finitely generated k-algebra ([MFK94, Thm. A.1.0], [Dol03, Thm. 3.2, 3.3], for counterexamples where R is not reductive, see [Nag60] and [Muk04]). **Thus, for the rest of the section, we fix G to be a reductive group.** Then it is natural to define the quotient to be the affine scheme

$$X//G := \mathrm{Spec}R^G.$$

However, in general, there is no one-to-one correspondence between the points of $X//G$ and the orbits of G.

Example 2.6. Consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$ by $(x, y) \mapsto (tx, t^{-1}y)$, then the orbits of this action are of three types:

(i) The origin $(0, 0)$ is an orbit consisting of one point.
(ii) For each nonzero $a \in k$, the hyperbola $xy = a$ is a single orbit.
(iii) The x-axis minus the origin and the y-axis minus the origin are orbits.

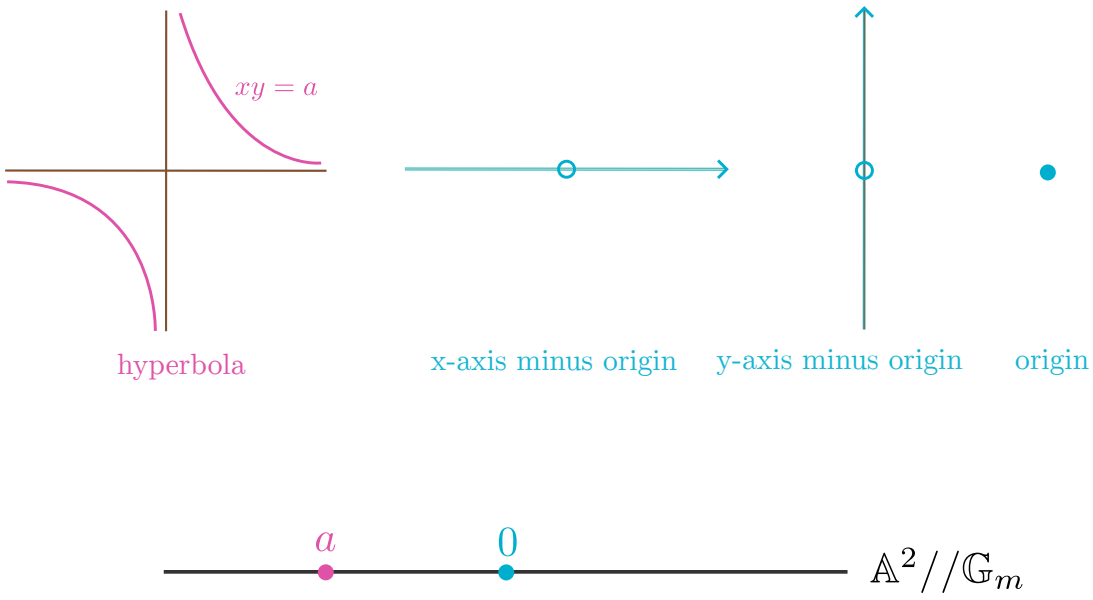


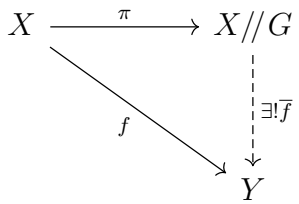
Figure 3: Orbits of $\mathbb{A}^2//\mathbb{G}_m$.

Using the construction before, we have the quotient

$$X//G = \text{Spec}k[x,y]^G = \text{Spec}[xy] = \mathbb{A}^1,$$

so the quotient map is $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ via $(x,y) \mapsto (xy)$. This map separates the orbits (ii) but cannot distinguish the three orbits of (i) and (iii), all of which map to zero.

The issue here is that the fibers of π are always closed in X, so two non-closed orbits will map to the same point via π. The GIT solution is to relax the condition (a) by identifying to a single point the three orbits which the invariants fail to separate, and use (b) to define a quotient in a universal categorical sense. More precisely speaking, the natural G-invariant projection $\pi : X \to X//G$ makes $X//G$ a *universal categorical quotient*, i.e., any G-invariant morphism $f : X \to Y$ factors through $X//G$:



With this definition, (a) is replaced by

(a') every point in $X//G$ corresponds to a unique closed orbit (and two orbits map to the same point in $X//G$ iff the intersection of their closures is non-empty).

In other words, $X//G$ is a *good quotient*. However, it is typically not a *geometric quotient*:

Definition 2.7. A morphism $\varphi : X \to Y$ is a *good quotient* for the action of G on X if

1. φ is G-invariant: $\varphi(g \cdot x) = \varphi(x)$ for all $g \in G, x \in X$.
2. φ is surjective.

3. If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism onto the G-invariant functions $\mathcal{O}_X(\varphi^{-1}(U))^G$.
4. If $W \subset X$ is a G-invariant closed subset of X, its image $\varphi(W)$ is closed in Y.
5. If W_1 and W_2 are disjoint G-invariant closed subsets, then $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.
6. φ is an affine map (i.e., the preimage of every affine open is affine).

Moreover, if the preimage of each point is a single orbit, then we say φ is a *geometric quotient*.

Theorem 2.8. Let G be a reductive group acting on an affine scheme X. Then the GIT quotient $\varphi : X \to X//G$ is a good quotient and moreover, $X//G$ is an affine scheme.

Proposition 2.9. Let G be an affine algebraic group acting on a scheme X and let $\varphi : X \to Y$ be a good quotient, then:

(a) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2}$ if and only if $\varphi(x_1) = \varphi(x_2)$.
(b) For each $y \in Y$, the preimage $\varphi^{-1}(y)$ contains a unique single orbit. In particular, if the action is closed (i.e., all orbits are closed), then φ is a geometric quotient.

The double slash notation $X//G$ is a reminder that the quotient is not necessarily a geometric quotient. To make it into a geometric quotient, we introduce the following notion:

Definition 2.10. We say $x \in X$ is *stable* if the orbit is closed in X and $\dim G_x = 0$ (or equivalently, $\dim G \cdot x = \dim G$). We let X^s denote the set of stable points.

Proposition 2.11. Let G be a reductive group acting on an affine scheme X and let $\varphi : X \to Y := X//G$ be the affine GIT quotient. Then $X^s \subset X$ is an open and G-invariant subset, $Y^s := \varphi(X^s)$ is an open subset of Y and $X^s = \varphi^{-1}(Y^s)$. Moreover, $\varphi : X^s \to Y^s$ is a geometric quotient.

Example 2.12. We can now calculate the stable set for the action $\mathbb{G}_m$ on $\mathbb{A}^2$ as before. The closed orbits are the conics $\{xy = a\}$ for $a \in \mathbb{A}^1 \setminus \{0\}$ and the origin, but the origin has a positive-dimensional stabilizer. Thus

$$X^s = \{(x, y) \in \mathbb{A}^2 : xy \neq 0\}.$$

In this example, it is clear why we need to insist that $\dim G_x = 0$ in the definition of stability: so that the stable set is open.

2.2.2 Projective GIT Quotient

Now, we want to extend the theory of affine GIT to construct GIT quotients for reductive group actions on projective varieties. The idea is that we would like to construct our GIT quotient by gluing affine GIT quotients. In order to do this, we would like to cover our scheme X by affine open subsets which are invariant under the group action and glue the affine GIT quotients of these affine open subsets of X. However, it may not be possible to cover all of X by such compatible open invariant affine subsets.

For a projective scheme X with an action of a reductive group G, there is no canonical way to produce an open subset of X that is covered by open invariant affine subsets. Instead, this will depend on a choice of an equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$, where G acts on $\mathbb{P}^n$ by a linear representation $G \to GL_{n+1}$.

Definition 2.13. Let X be a projective scheme with an action of an affine algebraic group G. A *linear G-equivariant projective embedding* of X is a group homomorphism $G \to GL_{n+1}$ and a G-equivariant projective embedding $X \hookrightarrow \mathbb{P}_n$. We will often simply say that the G-action on $X \hookrightarrow \mathbb{P}_n$ is *linear* to mean that we have a linear G-equivariant projective embedding of X.

Suppose we have a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$. Then the action of G on $\mathbb{P}^n$ lifts to an action of G on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$. Since the projective embedding $X \subset \mathbb{P}^n$ is G-equivariant, there is an induced action of G on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$ over $X \subset \mathbb{P}^n$.

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & \mathbb{A}^{n+1} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}^n \end{array}$$

More precisely, we have

$$\mathcal{O}(\mathbb{A}^{n+1}) = k[x_0, \ldots, x_n] = \bigoplus_{r \geq 0} k[x_0, \ldots, x_n]_r = \bigoplus_{r \geq 0} H^0(\mathbb{P}^n, \mathcal{O}_X(r)),$$

and if $X \subset \mathbb{P}^n$ is the closed subscheme associated to a homogeneous ideal $I(X) \subset k[x_0, \ldots, x_n]$, then $\tilde{X} = \mathrm{Spec} R(X)$ where $R(X) = k[x_0, \ldots, x_n]/I(X)$.

The k-algebras $\mathcal{O}(\mathbb{A}^{n+1})$ and $R(X)$ are graded by homogeneous degree, and as the G-action on $\mathbb{A}^{n+1}$ is linear, it preserves the graded pieces, so that the invariant subalgebra

$$\mathcal{O}(\mathbb{A}^{n+1})^G = \bigoplus_{r \geq 0} k[x_0, \ldots, x_n]_r^G$$

is a graded algebra and similarly, $R(X)^G = \oplus_{r \geq 0} R(X)_r^G$. By Nagata's theorem, $R(X)^G$ is finitely generated, as G is reductive. The inclusion $R(X)^G \hookrightarrow R(X)$ determines a rational morphism of projective varieties

$$X \dashrightarrow \mathrm{Proj} R(X)^G$$

whose indeterminancy locus is the closed subscheme of X defined by the (irrelevant) homogeneous ideal $R(X)_+^G := \oplus_{r>0} R(X)_r^G$ by construction.

Definition 2.14. For a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}_n$, the *nullcone* N is defined to be the closed subscheme of X defined by the homogeneous ideal $R(X)_+^G$ in $R(X)$. We define the semistable set $X^{ss} = X - N$ to be the open subset of X given by the complement of the nullcone. More precisely, $x \in X$ is *semistable* if there exists a G-invariant homogeneous function $f \in R(X)_r^G$ for $r > 0$ such that $f(x) \neq 0$. By construction, the semistable set is the open subset which is the domain of the rational map

$$X \dashrightarrow \mathrm{Proj} R(X)^G.$$

We call the morphism $X^{ss} \to X//G := \mathrm{Proj} R(X)^G$ the *GIT quotient* of this action.

Theorem 2.15. For a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$, the GIT quotient $\varphi : X^{ss} \to X /\!/ G$ is a good quotient of the G-action on the open subset X^{ss} of semistable points in X. Furthermore, $X /\!/ G$ is a projective scheme.

The presence of non-closed orbits in the semistable locus prevents the good quotient $\varphi : X^{ss} \to X//G$ from being a geometric quotient. We can now ask if there is an open subset X^s of X^{ss} on which this quotient becomes a geometric quotient. For this, we want the action to be closed on X^s. This, similar to the affine case, motivates the definition of stability.

Definition 2.16. Consider a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$. We say a point $x \in X$ is

1. *stable* if $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)^G_+$ such that $x \in X_f$ and the action of G on X_f is closed.
2. *unstable* if it is not semistable.

We denote the set of stable points by X^s.

Theorem 2.17. Following the notation above. Let $\varphi : X^{ss} \to Y := X//G$ denote the GIT quotient. There is an open subscheme $Y^s \subset Y$ such that $\varphi^{-1}(Y^s) = X^s$ and that the GIT quotient restricts to a geometric quotient $\varphi : X^s \to Y^s := X/G$.

$$\begin{array}{ccc} X^{ss} & \longrightarrow & X//G \\ \downarrow & & \downarrow \\ X^s & \longrightarrow & X/G \end{array} \qquad \begin{array}{l} \text{good quotient} \\ \\ \text{geometric quotient} \end{array}$$

Note that when $X//G$ is a geometric quotient, we write X/G with a single slash to emphasize the fact that it is an orbit space. Furthermore, since $X^{ss}//G = X//G$ is a projective scheme, it is necessarily compact. Hence, it can be seen as the compactification of X^s/G, called the *GIT compactification*.

2.2.3 The Hilbert-Mumford Criterion

We also wish to determine stable and semistable points of a space. To do so, it is necessary to study the closure of the orbits. One way to do so is by using a 1-parameter subgroup of G. As usual, let G be a reductive group which acts on a projective scheme $X \subset \mathbb{P}^n$ linearly.

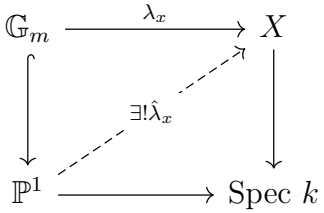
Definition 2.18. A *1-parameter subgroup* (1-PS) of G is a non-trivial group homomorphism $\lambda : \mathbb{G}_m \to G$.

Fix $x \in X$ and a 1-PS $\lambda : \mathbb{G}_m \to G$. Then we let $\lambda_x : \mathbb{G}_m \to X$ be the morphism given by

$$\lambda_x(t) = \lambda(t) \cdot x.$$

There is a natural embedding of $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{P}^1$ via $t \mapsto [1 : t]$. Since X is projective, it is proper over Spec$k$, so the morphism $\lambda_x : \mathbb{G}_m \to X$ extends uniquely to a morphism

$\hat{\lambda}_x : \mathbb{P}^1 \to X$.



In particular, we define

$$\lim_{t\to 0} \lambda(t)\cdot x := \hat{\lambda}_x([1:0]) \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)\cdot x := \hat{\lambda}_x([0:1]).$$

In fact, we can focus on either of them, as

$$\lim_{t\to\infty} \lambda(t)\cdot x = \lim_{t\to 0} \lambda^{-1}(t)\cdot x.$$

Let $y := \lim_{t\to 0} \lambda(t)\cdot x$, then y is fixed by the action of $\lambda(\mathbb{G}_m)$, therefore, on the fiber over y of the line bundle $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^n}(1)|_X$, the group $\lambda(\mathbb{G}_m)$ acts by a character $t \mapsto t^r$.

Definition 2.19. The Hilbert-Mumford weight of the action of the 1-PS λ on $x \in X$ is defined to be

$$\mu^{\mathcal{O}(1)}(\lambda, x) := r$$

where r is the weight of $\lambda(\mathbb{G}_m)$ on the fiber $\mathcal{O}(1)_y$ over $y := \lim_{t\to 0} \lambda(t)\cdot x$.

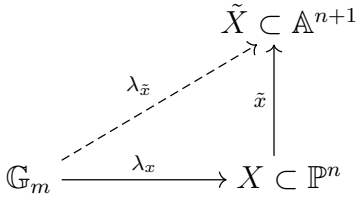
This allows us to study the (semi)stability of points.

Theorem 2.20 (Hilbert-Mumford Criterion)**.** Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then, for $x \in X$, we have

1. $x \in X^{ss} \iff \mu(x,\lambda) \geq 0$ for all 1-PSs λ of G,
2. $x \in X^{s} \iff \mu(x,\lambda) > 0$ for all 1-PSs λ of G.

However, from this definition, the Hilbert-Mumford weight is difficult to compute. So we reformulate it in terms of the weights for the action on the affine cone.

Let $\mathbb{A}^{n+1}$ be the affine cone over $\mathbb{P}^n$. Pick a non-zero lift $\tilde{x} \in \tilde{X}$ of $x \in X$. Then we can consider morphism $\lambda_{\tilde{x}} : \mathbb{G}_m \to \tilde{X}$ defined as



which may no longer extend to $\mathbb{P}^1$, as $\tilde{X}$ is not proper. If it extends to zero or infinity, we denote the limits by

$$\lim_{t\to 0} \lambda(t)\cdot \tilde{x} \quad \text{or} \quad \lim_{t\to\infty} \lambda(t)\cdot \tilde{x}.$$

The action of the 1-PS $\lambda(\mathbb{G}_m)$ on the affine cone $\mathbb{A}^{n+1}$ is linear, so we can choose a basis $e_0, \ldots, e_n$ of k^{n+1} such that

$$\lambda(t)\cdot(e_0,\ldots,e_n) = (t^{r_0}e_0, \ldots, t^{r_n}e_n) \text{ for } r_i \in \mathbb{Z}.$$

For $x \in X$, we can pick $\tilde{x} \in \tilde{X}$ lying about x and write $\tilde{x} = \sum_{i=0}^{n} x_i e_i$ with respect to this basis, then

$$\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} e_i.$$

We let λ-weights of x to be defined as $\{r_i : x_i \neq 0\}$. Note that this does not depend on the choice of lift of $\tilde{x}$.

Definition 2.21. The Hilbert-Mumford weight of x at λ is defined as

$$\mu(x, \lambda) := -\min\{r_i : x_i \neq 0\}.$$

Lemma 2.22. The two definitions of the Hilbert-Mumford weight agree:

$$\mu^{\mathcal{O}(1)}(x, \lambda) = \mu(x, \lambda).$$

Example 2.23. We compute the (semi)stability of binary forms in $\mathbb{P}^n$ for $n = 2m$ even. By the Hilbert-Mumford criteria, $f \in \mathbb{P}^n$ is semistable if and only if $\mu(\lambda, f) \geq 0$. We choose a basis, and let $f(x, y) = \sum_{i=1}^{n} a_i x^i y^{n-i}$. SL_2 is rank 2, so any 1-parameter subgroup $\lambda : \mathbb{C}^* \to SL_2$ can be diagonalized to $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, meaning that it is modeled by the 1-parameter subgroup defined by $\lambda(t)(x) = tx$ and $\lambda(t)y = t^{-1}y$. Then

$$\lambda(t)f(x, y) = \sum_{i=1}^{n} a_i t^i x_i t^{-(d-i)} y^{d-i} = \sum_{i=1}^{d} a_i t^{2i-d} x^i y^{d-i},$$

so $r_i = 2i - d$. Then we have

$$\mu(\lambda, x) = -\min\{2i - d \,|\, a_i \neq 0\} = \max\{d - 2i \,|\, a_i \neq 0\}.$$

Let i_0 be the minimum i such that $a_i \neq 0$, then $\mu(\lambda, x) = d - 2i_0$. Now, since the SL_2-action is transitive, we can move any root to x, so this result holds for all roots. We conclude that f is semistable if and only if all roots have multiplicity $\leq m$. A similar computation shows that f is stable if and only if all roots have multiplicity $< m$. Hence, the f is strictly semistable if and only if all roots have multiplicity m.

2.3 Curves

Let us recall some basic curve theory.

Definition 2.24. A *(complex) curve* is a one-dimensional variety $C \to \mathrm{Spec}(\mathbb{C})$. In other words it is a separated, reduced scheme of finite type over $\mathrm{Spec}(\mathbb{C})$ such that all irreducible components have dimension 1.

Definition 2.25. Let C be a complex projective curve.

(a) If C is smooth, its *geometric genus* is defined to be

$$p_g(C) = h^{1,0}(C) = \dim H^0(C, \Omega_C)^1,$$

where Ω_C^1 is the contangent line bundle of C. For a singular curve C, its geometric genus is defined to be the geometric geuns of its normalization.

(b) The *arithmetic genus* of C is defined as

$$p_a(C) = 1 - \chi(C, \mathcal{O}_C) = 1 - \dim H^0(C, \mathcal{O}_C) + \dim H^1(C, \mathcal{O}_C).$$

Let C be a smooth, projective, irreducible curve. Let $C(\mathbb{C})$ denote its $\mathbb{C}$-points with complex topology.

Definition 2.26. Let C be a complex curve.

(a) A closed point $q \in C$ is a *node* if there exists a neighborhood of $q \in C(\mathbb{C})$ which is complex-analytically isomorphic to a neighborhood of the origin in the locus $\{(x, y) : xy = 0\} \subset \mathbb{C}^2$.

(b) The curve C is called *nodal* if every closed point $q \in C$ is either a smooth point or a node.

Smooth curves of genus g degenerate to nodal curves of genus g. However, it turns out that there are too many nodal curves of genus g to obtain a compact moduli space. Hence restrict ourselves to so-called *stable* curves.

Definition 2.27. A connected, nodal, complex projective curve C is called *stable* (respectively *semistable*) if the group

$$\mathrm{Aut}(C) = \{\varphi : C \to C : \varphi \text{ isomorphism}\}$$

of its automorphisms is finite (resp. reduced). Moreover, given $p_1, \ldots, p_n \in C$ distinct smooth points of C, we say that $(C, p_1, \ldots, p_n)$ is stable if the group

$$\mathrm{Aut}(C) = \{\varphi : C \to C : \varphi \text{ isomorphism with } \varphi(p_i) = p_i\}$$

of automorphisms of C fixing all p_i is finite (resp.).

Proposition 2.28. Following the notation above, $(C, p_1, \ldots, p_n)$ is stable (resp. semistable) if and only if every irreducible component $\widetilde{C}_\nu \subset \widetilde{C}$ of the normalization of C satisfies

- $\widetilde{C}_\nu$ has genus 0 and contains at least 3 special points (resp. 2 special points), i.e., preimages of nodes of C or markings p_i, or
- $\widetilde{C}_\nu$ has genus 1 and contains at least 1 special point, or
- $\widetilde{C}_\nu$ has genus at least 2.

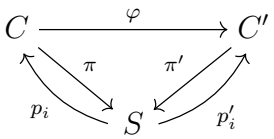
Definition 2.29. Given $g, n \geq 0$, an *n-pointed family of (smooth/stable) genus g curves* over a scheme S is a tuple

$$(\pi : C \to S; p_1, \ldots, p_n : S \to C)$$

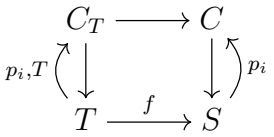
where

- π is a (smooth/flat), proper, surjective, finitely presented morphism of schemes such that the fiber C_s over any geometric point $s \in S$ is a (smooth/stable), projective, connected curve of arithmetic genus g,
- the morphisms $p_1, \ldots, p_n$ are pairwise disjoint sections of π, with image in the smooth locus of π.

We say that a second family $(C'/S; p'_1, \ldots, p'_n)$ is *isomorphic* to $(C/S; p_1, \ldots, p_n)$ if there exists an isomorphism $\varphi : C \to C'$ over S such that $\varphi \circ p_i = p'_i$, i.e., such that the diagram



commutes. Given a map $f : T \to S$, we define the *pullback* of $(C/S; p_1, \ldots, p_n)$ under f to be the family $(C_T/T; p_{1,T}, \ldots, p_{n,T})$ for the fiber product $C_T = C \times_S T$with the induced sections $p_{i,T} = (p_i \circ f) \times id_T$.



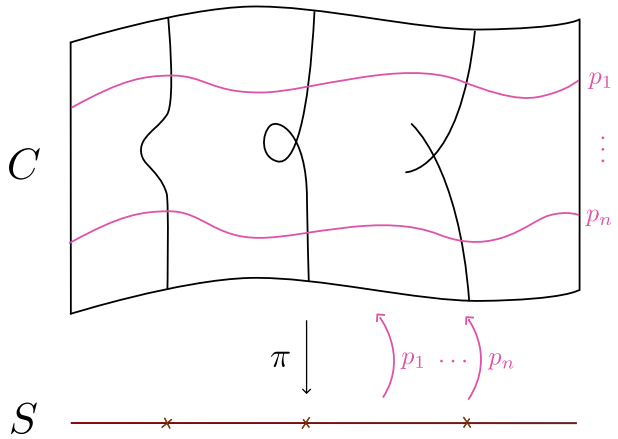


Figure 4: A family of stable curves.

Definition 2.30. A (projective regular integral) genus g curve C is *hyperelliptic* if it admits a double cover (i.e., degree 2, necessarily finite, morphism) to $\mathbb{P}^1$. If $\pi^{-1}(p)$ is a single point, we say p is a *branch point*, and $\pi^{-1}(p)$ is a *ramification point* of $\pi : C \to \mathbb{P}^1$.

Theorem 2.31 (hyperelliptic Riemann-Hurwitz formula)**.** Let $\pi : C \to \mathbb{P}^1$ be a double cover by a projective regular irreducible genus g curve over . Then π have $2g + 2$ branch points.

A property of hyperelliptic cruves is that a genus ≥ 2 curve can be hyperelliptic in only one way:

Proposition 2.32. Any curve C of genus at least 2 admits at most one double cover pf $\mathbb{P}^1$. More precisely, if $\mathcal{L}$ and $\mathcal{M}$ are two degree two line bundles yielding maps $C \to \mathbb{P}^1$, then $\mathcal{L} \cong \mathcal{M}$.

Hence a hyperelliptic curve of genus $g \geq 2$ correspond to precisely $2g+2$ distinct points on $\mathbb{P}^1$ modulo S_{2g+2}, and modulo automorphisms of $\mathbb{P}^1$. This defines an isomorphism between $\mathcal{B}_{2g+2}$, the moduli space of binary forms of degree $2g + 2$ with no multiple roots, and $\mathcal{H}_g$, the moduli space of smooth hyperelliptic curves, by viewing a binary form of degree n as $\mathbb{P}^1$ with n distinct points.

3 Pencils of Quadrics

In this section we build up the general theory of pencils of quadrics. We refer the reader to [OSG20] for more background on pencils. The results from now on are from [AL00] and [AL01].

3.1 Pencils of Quadrics

Assume that $A, B \in \mathbb{C}^{(n+1)\times(n+1)}$ are two symmetric matrices which are not scalar multiples of each other. With $x = (x_0, \ldots, x_n)$, two quadrics A and B in $\mathbb{P}^n$ can be defined by

$$A : x^T Ax = 0 \text{ and } B : x^T Bx = 0. \tag{1}$$

Definition 3.1. Let A and B be two quadrics in $\mathbb{P}^n$ with equation (1), then the *pencil* $\mathcal{P}$ *of quadrics* spanned by A and B contains quadrics given by

$$\mathcal{P} : x^T(\lambda A + \mu B)x = 0 \quad (\lambda : \mu) \in \mathbb{P}^1.$$

A quadric is called *regular* if its matrix is full rank. Otherwise it is called *singular*. The singular quadrics in a pencil are called *cones*. In particular, a *d-cone* is a cone with a singular locus of (projective) dimension d.

Let $N := \frac{1}{2}n(n+3)$, we claim that $\mathbb{P}^N$ can be considered as the space of quadrics in $\mathbb{P}^n$. Indeed, a quadric is determined by an $(n+1) \times (n+1)$ symmetric matrix, which is determined by $1 + 2 + \cdots + (n+1) = \frac{1}{2}(n+1)(n+2)$ entries. This corresponds to a projective space of dimension $\frac{1}{2}(n+1)(n+2) - 1 = \frac{1}{2}n(n+3)$. A pencil of quadrics in $\mathbb{P}^n$ is a line in $\mathbb{P}^N$, so the space of pencils of quadrics is the Grassmannian $Gr(2, N+1)$, the set of all 2-dimensional linear subspaces of $N+1$ dimensional vector space. A pencil $\mathcal{P} \in Gr(2, N+1)$ is usually given as $\{\lambda A + \mu B \,|\, (\lambda : \mu) \in \mathbb{P}^1\}$. By abuse of notation we write

$$\mathcal{P} = \lambda A + \mu B.$$

For any $(\lambda_0 : \mu_0) \in \mathbb{P}^1$, we denote $Q_{(\lambda_0:\mu_0)}$ or inhomogeneously (with a slight abuse of notation) Q_{λ_0} the quadric in $\mathbb{P}^n$ given by the matrix $l_0 A + \mu_0 B$.

Definition 3.2. The discriminant of the pencil $\mathcal{P}$ is the binary form in λ and μ:

$$\Delta = \Delta(\lambda, \mu) := \det(\lambda A + \mu B).$$

Note that the discriminant $\Delta(\mathcal{P})$ depends on the choice of the matrices A and B. The roots of Δ, however, are uniquely determined up to an isomorphism of $\mathbb{P}^1$. In particular the multiplicities of the roots are uniquely determined. A pencil of quadrics is called *general* if $\Delta(\mathcal{P})$ admits no multiple roots, in other words, if it contains no singular quadrics. Note that Δ is not identically zero if and only if Δ is a binary form of degree $n+1$ if and only if the general quadric of the pencil is nondegenerate.

The (distinct) roots of the discriminant precisely correspond to the singular quadrics in the pencil. We are interested in understanding their structure. One way to do so is via its Segre symbol.

For a root α_i of Δ of multiplicity e_i, we know that it correspond to the singular quadric hypersurface given by a cone over a smooth quadric. If α_i is not only a zero of $\det(\lambda A + \mu B)$, but also of all its subdeterminants of size $n - h_i + 2$ for $h_i \geq 2$ (where h_i is chosen maximally), then the dimension of the vertex of the cone is precisely $h_i - 1$. Note that the case $h_i = 1$ corresponds to the generic situation, where the vertex is just a point.

Then define l_j^i to be the minimum multiplicity of the root α_i for the set of subdeterminants of size $n + 1 - i$ for $0, 1, \ldots, h_i - 1$. In other words, let Det_{n+1-i} denote the set of all subdeterminants of $\mathcal{P}$ of size $n + 1 - i$. Then

$$l_j^i := \min_{d(\lambda,\mu)\in \mathrm{Det}_{n+1-i}} \{k : (\lambda - \alpha)^k \text{appears in } d(\lambda, \mu)\}.$$

Then $l_j^i \geq l_j^{i+1}$, so we can set $e_i^j := l_i^j - l_{i+1}^j > 0$. Then $\Delta(\lambda, \mu)$ can be factored as

$$\Delta(\lambda, \mu) = (\lambda - \alpha_i)^{e_0^i} \cdots (\lambda - \alpha_i)^{e_{h_i-1}^i} f_i(\lambda)$$

for some $f_i(\alpha) \neq 0$.

Definition 3.3. The polynomials $(\lambda - \alpha_i)^{e_j^i}$ are the *elementary divisors* associated to α_i, and the exponents e_i are the *characteristic numbers*. The Segre symbol of a pencil is

$[(e_0^1, \ldots, e_{h_0-1}^1), (e_0^2, \ldots, e_{h_1-1}^2), \ldots, (e_0^r, \ldots, e_{h_r-1}^r)]$ is called the *Segre symbol* of the pencil $\mathcal{P}$. The parentheses are omitted if $h_i = 1$.

Segre symbols are useful in the classification of pencils: two pencils of quadrics $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{P}^n$, whose discriminants have roots exactly at $(1 : \alpha_i)$ and $(1 : \beta_i)$ are isomorphic, that is, projectively equivalent in $\mathbb{P}^n$, if and only if they have the same Segre symbol and there is an automorphism of $\mathbb{P}^1$ taking $(1 : \alpha_i)$ to $(1 : \beta_i)$ for all i. This can be used to define a normal form for the pencils $\mathcal{P}$ whose discriminant is not identically zero ([HP94, p. 280]). Let $(\lambda - \alpha_1)^{e_1}, (\lambda - \alpha_2)^{e_2}, \ldots, (\lambda - \alpha_r)^{e_r}$ denote the elementary divisors of $\mathcal{P}$. Then the coordinates can be chosen in such a way that $\mathcal{P} = \lambda A + \mu B$ with $A = diag(P_1, \ldots, P_r)$ and $B = diag(Q_1, \ldots, Q_r)$ are the diagonal block matrices with blocks

$$P_i = \begin{pmatrix} & & & a_i \\ & & a_i & 1 \\ & \iddots & \iddots & \\ a_i & 1 & & \end{pmatrix} \quad \text{and } Q_i = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

of size e_i.

The parentheses in the Segre symbol correspond one-to-one to the cones in the pencil. Suppose Q_λ is a cone in the pencil corresponding to $(e_0, \ldots, e_d)$. We call Q_λ a *cone of type* $(e_0, \ldots, e_d)$. The quadric Q_λ is then a d-cone and the corresponding root in the discriminant Δ is a root of multiplicity $e := \sum_{i=1}^d e_i$. By a slight abuse of notation we call Q_λ a *d-cone of multiplicity e.*

Example 3.4. Let us provide some insights into the Segre symbol by doing a sample

computation. The matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

define a pencil

$$\mathcal{P}(\lambda, \mu) = \lambda A + \mu B = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & \lambda \\ 0 & 0 & \lambda & \mu \end{pmatrix}.$$

We have $\Delta(\mathcal{P}) = \lambda^2(\mu^2 - \lambda^2) = \lambda^2(\mu + \lambda)(\mu - \lambda)$, so the roots are $(0:1), (1:-1)$ and $(1:1)$.

For the root $(0:1)$ which corresponds to λ, we note that λ is a root of all 3×3 minors, so $h_i = 2$. Now let us compute l_j :

- when $j = 0$, we have the entire matrix, which corresponds to $\Delta(P) = \lambda^2(\mu+\lambda)(\mu-\lambda)$. Since the multiplicity of λ is 2, we have $l_0 = 2$.
- when $j = 1$, we consider 3×3 subdeterminants. The only nonzero subdeterminants are $\begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{vmatrix} = \lambda^2\mu$ and $\begin{vmatrix} \lambda & 0 & 0 \\ 0 & \mu & \lambda \\ 0 & \lambda & \mu \end{vmatrix} = \lambda(\mu + \lambda)(\mu - \lambda)$, meaning that the minimum multiplicity of λ is 1, hence $l_1 = 1$.
- when $j = 2$, we consider 2 nonzero subdeterminants $\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \lambda^2$ and $\begin{vmatrix} \mu & \lambda \\ \lambda & \mu \end{vmatrix} = (\mu + \lambda)(\mu - \lambda)$, hence $l_2 = 0$.

This gives us $(e_0^1, e_1^1) = (1, 1)$ for the root $(0:1)$. Using a similar logic for the other two roots, we get that the Segre symbol of this pencil is $[(1, 1), 1, 1]$. The elementary divisors here are $(\lambda - \alpha), (\lambda - \alpha), (\lambda - \beta), (\lambda - \gamma)$ where α, β, γ refer to the three roots.

Compare it with the pencil

$$\mathcal{Q}(\lambda, \mu) = \begin{pmatrix} \mu & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & \mu & \lambda \\ 0 & 0 & \lambda & \mu \end{pmatrix}.$$

It has discriminant $\Delta(\mathcal{Q}) = -\lambda^2(\mu - \lambda)(\mu + \lambda)$ with roots $(0:1), (1:-1)$, and $(1:1)$.

For the root $(0:1)$, we know $l_0 = 2$ since λ has multiplicity 2 in the discriminant. But now $l_1 = 0$ since the non-zero 3×3 subdeterminants are $-\mu(\lambda^2)$ and $(\mu - \lambda)(\mu + \lambda)$, and in the latter subdeterminant λ appears nowhere. Hence the Segre symbol of this pencil is $[2, 1, 1]$. The elementary divisors here are $(\lambda - \alpha)^2, (\lambda - \beta), (\lambda - \gamma)$ where α, β, γ refer to the three roots.

Geometrically, Segre symbol measures the type of singularity in the pencil. The pencil $\mathcal{P}$ has Segre symbol $[2, 1, 1]$, meaning that the characteristic numbers are $(2, 0), (1, 0)$ and $(1, 0)$, so its singular quadrics are three 0-cones corresponding to roots of multiplicity $2, 1, 1$ respectively. Compare it to the pencil $\mathcal{Q}$ with Segre symbol $[(1, 1), 1, 1]$, meaning its characteristics are $(1, 1), (1, 0), (1, 0)$, i.e., its singular quadrics are a 1-cone (corresponding

to a double root) and two 0-cones (corresponding to single roots).

3.2 (Semi-)stability of Pencils of Quadrics

We call a pencil of quadrics *general* if its discriminant admits no multiple root. A pencil of quadrics in $\mathbb{P}^n$ is a point in $Gr(2, N+1)$. There is an action of SL_{n+1} on $Gr(2, N+1)$ by change of coordinates

$$\begin{aligned} SL_{n+1}(\mathbb{C}) \times Gr(2, N+1) &\to Gr(2, N+1) \\ (T, P = \lambda A + \mu B) &\mapsto \lambda TAT^t + \mu TBT^t. \end{aligned}$$

The Grassmannian $Gr(2, N+1)$ can be thought as a projective variety via the Plücker embedding, so the action by $SL_{n+1}(\mathbb{C})$ gives a notion of stability of semistability of pencils of quadrics $\mathbb{P}^n$. By GIT theory, this gives rise to $\mathcal{P}_n := Gr(2, N+1)^s/SL_{n+1}$ the moduli space of general pencils of quadrics in $\mathbb{P}^n$ and $\overline{\mathcal{P}}_n := Gr(2, N+1)^{ss}//SL_{n+1}$ its compactification. To better understand the spaces, we need to know what the stable and semistable points look like. The aim of this section is to give a (semi)-stability condition on pencils of quadrics:

Theorem 3.5.
1. A pencil $P \in Gr(2, N+1)$ is semistable if and only if its discriminant admits no root of multiplicity $> n/2$.
2. A pencil $P \in Gr(2, N+1)$ is stable if and only if its discriminant admits no multiple root.

The idea of the proof is to reduce the (semi)stability of a pencil to the (semi)stability of a principal PGL_2-bundle over an open set of $Gr := Gr(2, N+1)$. For the proof we need some preliminaries.

Note that the discriminant as defined is a map $Gr \to \mathbb{P}^n$ from the space of pencils to the space of binary forms. However, this map is only defined on the subset of Gr which contains pencils of nonzero determinant. More precisely, consider the affine space $V = \mathbb{C}^{N+1}$ of symmetric $(n+1) \times (n+1)$ matrices and let $\mathbb{P}^(V \times V)$ denote the projective space of lines in $V \times V$, we define

$$S_n := \{(A, B) \in \mathbb{P}(V \times V) \mid \text{A and B are linearly independent.}\}$$

A pencil of quadrics is called *nondegenerate* if it contains a nondegenerate quadric, that is, the matrix associated to the quadric has non-zero determinant, or equivalently if the general quadric of the pencil is nondegenerate. Let W_n denote the open subset S_n whose corresponding pencil is nondegenerate, that is,

$$W_n := \{(A, B) \in S_n \,|\, rk(\lambda A + \mu B) = n+1 \text{ for general } (\lambda : \mu).\}$$

Let $\alpha = (\alpha_{ij}) \in_2$, then the group PGL_2 (or SL_2) acts on S_n by

$$\alpha \circ (A, B) := (\alpha_{11}A + \alpha 21B, \alpha_{21}A + \alpha 22B).$$

Note that the projection

$$\pi : S_n \to Gr$$

has the structure of a principal-PGL_2-bundle. In particular we have $Gr \cong S_n/PGL_2$. The action of SL_{n+1} on Gr lifts to an action on S_n (hence also W_n) which commutes with the action of PGL_2. So we have the equivalence $Gr^{ss}//SL_{n+1} = W_n^{ss}//(SL_{n+1} \times PGL_2)$. Note that $PGL_2 = SL_2/\{\pm I\}$, but $\{\pm I\}$ acts on the pencils trivially, so the two actions are equivalent. For simplicity, we work with SL_2 for now.

Let $\mathbb{P}_{n+1}$ denote the space of binary forms of degree $n+1$. The group PGL_2 acts on $\mathbb{P}_{n+1}$ via change of coordinates

$$\alpha \circ f_{n+1}(\lambda, \mu) := f(\alpha_{11}\lambda + \alpha_{12}\mu, \alpha_{21}\lambda + \alpha_{22}\mu).$$

Define $D(A, B) := \det(\lambda A + \mu B)$. Then we have the following.

Lemma 3.6. The map $D : W_n \to \mathbb{P}^{n+1}$ is a proper (i.e., it is continuous and preimage of a compact set is compact, in the analytic topology), subjective morphism equivariant with respect to the action of $SL_{n+1} \times SL_2$ where SL_{n+1} acts trivially on $\mathbb{P}_n$.

Proof. Recall that the action is equivariant means that for any $T \in SL_{n+1}$ and $\alpha \in SL_2$, we have $D((T, \alpha) \cdot (A, B)) = (T, \alpha) \cdot D(A, B)$. In other words, the following diagram commutes:

$$\begin{array}{ccc} (SL_{n+1} \times SL_2) \times W_n & \xrightarrow{(id,D)} & (SL_{n+1} \times SL_2) \times \mathbb{P}^n \\ \downarrow{\scriptstyle \sigma_{W_n}} & & \downarrow{\scriptstyle \sigma_{\mathbb{P}^n}} \\ W_n & \xrightarrow[D]{} & \mathbb{P}^n \end{array}$$

where the vertical arrows correspond to the actions of $SL_{n+1} \times SL_2$ on W_n and $\mathbb{P}^n$. The commutativity of the diagram follows from a simple computation using the fact that $T \in SL_{n+1}$ so $\det(T) = \det(T^t) = 1$.

The surjectivity of D follows from the theory of normal forms of pencils in the previous section. To show continuity, note that the preimage of a closed set in $\mathbb{P}^{n+1}$ is given by equations in $\mathbb{P}(V \times V)$ and hence is closed, and furthermore it is contained in W_n. Let $K \subset \mathbb{P}^{n+1}$ be compact, then $D^{-1}(K)$ is bounded because K only contains nonzero forms, so its preimage cannot diverge to a degenerate pencil. Hence $D^{-1}(K)$ is compact. □

Proof of Theorem 3.5. Note that W_n is a principal PGL_2-bundle over an open set of Gr. It follows that the (semi)-stability of a point $(A, B) \in W_n$ with respect to the group $SL_{n+1} \times SL_2$ is equivalent to the (semi)-stability of the pencil $\mathcal{P} = \pi(A, B)$ with respect to SL_{n+1}. Hence we can prove the theorem on the level of W_n.

(a): We claim that (A, B) is unstable in W_n if and only if $D(A, B)$ is unstable in $\mathbb{P}^{n+1}$. Indeed, write $G := SL_{n+1} \times SL_2$. Assume (A, B) is unstable, then $0 \in \overline{G \cdot (A, B)}$. Recall that the map D is continuous, so $D(\overline{A}) \subset \overline{D(A)}$ for all subsets A in the domain. Applying this to the orbit, we have

$$0 \in D(\overline{G \cdot (A, B)}) \subset \overline{D(G \cdot (A, B))} = \overline{G \cdot D(A, B)}$$

where the last equality is due to the equivariance of D, which preserves orbits. Since 0 is in the closure of the orbit of $D(A, B)$, $D(A, B)$ is unstable in $\mathbb{P}^{n+1}$. Conversely, assume $D(A, B)$ is unstable in $\mathbb{P}^{n+1}$, then by [AM99, Section 3], (A, B) is unstable in W_n. So we

have now that (A, B) is unstable in W_n if and only if $D(A, B)$ is unstable in $\mathbb{P}^{n+1}$, which, as shown in example 2.23, is true if and only if it admits a root of multiplicity $> n/2$.

(b): *if $D(A, B)$ admits no multiple roots, then $(A, B) \in W_n$ is stable:*

Recall that W_n is stable if its orbit is closed and it has finite stabilizer. Assume $D(A, B)$ admits no multiple roots, then by example 2.23, $D(A, B) \in \mathbb{P}^n$ is stable, so its orbit $O(D(A, B)) \subset \mathbb{P}^{n+1}$ is closed. Since D is proper, its preimage $D^{-1}(O(D(A, B))) \subset W_n$ is also closed. But here each pencil is uniquely determined by its discriminant, so $O(A, B) = D^{-1}(O(D(A, B))) \subset W_n$ is closed. Furthermore, stabilizers of (A, B) are by definition $\{(T, \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in SL_{n+1} \times SL_2 : (T(aA + cB)T^t, T(bA + dB)T^t) = (A, B)\}$, which, after counting the number of constraints, can be shown to be finite.

If $\Delta(P)$ admits multiple roots, then $P \in Gr(2, N + 1)$ is not stable: All pencils whose discriminant admits multiple roots are in the closure of pencils of type $[2, 1, \ldots, 1]$ and $[(1, 1), 1 \ldots, 1]$. So it suffices to show pencils of these two types are not stable.

A pencil of type $[2, 1, \ldots, 1]$ is strictly semistable: A normal form of such pencil, according to section 3.1, is $\mathcal{P} = \lambda A + \mu B$, where

$$A = \begin{pmatrix} \lambda_0 & & & & & & \\ & \lambda_1 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_{n-2} & & & \\ & & & & 0 & \lambda_{n-1} \\ & & & & \lambda_{n-1} & 1 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

This means the equations defining $\mathcal{P}$ are

$$XAX^t = \lambda_0 X_0^2 + \lambda_1 X_1^2 + \cdots + \lambda_{n-2} X_{n-2}^2 + X_n^2 + 2\lambda_{n-1} X_{n-1} X_n$$
$$XBX^t = X_0^2 + X_1^2 + \cdots + X_{n-2}^2 + 2X_{n-1} X_n$$

with $\lambda_i \neq 0, 1, \lambda_i$ for $i \neq j$. Recall that by Theorem 2.20, $\mathcal{P}$ is not stable if and only if there exists a 1-PS λ such that $\mu(x, \lambda) \leq 0$ with respect to the Plücker embedding $Gr \to \mathbb{P}^M$. The nonzero Plücker coordinates of $\mathcal{P} = \lambda A + \mu B$ are the nonzero 2×2 subdeterminants m_{ij}, $(0 \leq ij \leq n)$ of the matrix with

$$\begin{pmatrix} \lambda_0 X_0^2 & \lambda_1 X_1^2 & \ldots & \lambda_{n-2} X_{n-2}^2 & X_n^2 & 2\lambda_{n-1} X_{n-1} X_n \\ X_0^2 & X_1^2 & \ldots & X_{n-2}^2 & 0 & 2X_{n-1} X_n \end{pmatrix}.$$

Let $\lambda(t) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & t^{-1} & \\ & & & & & t \end{pmatrix}$, then

$$\lambda(t) \cdot XAX^t = \lambda_0 X_0^2 + \lambda_1 X_1^2 + \cdots + \lambda_{n-2} X_{n-2}^2 + t^2 X_n^2 + 2\lambda_{n-1} X_{n-1} X_n,$$
$$\lambda(t) \cdot XBX^t = XBX^t.$$

Viewing it as a point in $\mathbb{P}^M$ via Plücker coordinates, we get that

$$\mu(\mathcal{P}, \lambda) = -\min\{0, 2\} = 0.$$

Hence $\mathcal{P}$ is not stable.

Any pencil of type $[(1,1), 1, \ldots, 1]$ *is not stable:*

A normal form of such a pencil is $\mathcal{P} = \lambda A + \lambda B$ with

$$A = \begin{pmatrix} \lambda_0 & & & & & & \\ & \lambda_1 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_{n-2} & & & \\ & & & & \lambda_{n-1} & & \\ & & & & & \lambda_{n-1} & \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \end{pmatrix}.$$

We claim the stabilizers are not finite, for instance the matrices

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \cos t & \sin t \\ & & & -\sin t & \cos t \end{pmatrix}$$

lie in SL_{n+1} and stabilize A and B with respect to the SL_{n+1} conjugation action. □

Remark. We corrected the typographical errors in the proof of Theorem 3.1(b) in [AL00], which included erroneous entries of the matrix corresponding to the Plücker coordinates, leading to incorrect Hilbert-Mumford weights. We clarified the argumeng by providing explicit computations to the Hilbert-Mumford weights.

4 Pencils & Binary Forms

Let V_d be the space of binary forms of degree d. Then there is an action

$$SL_2 \times V_d \to V_d$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, f(x,y) \to f(ax+by, cx+dy).$$

This gives rise to a notion of stability and semistability of binary forms. Let $\mathcal{B}_n$ (respectively $\overline{\mathcal{B}}_n$) denote the moduli space of stable (respectively semistable) binary forms of degree n. The goal of this section is to show $\overline{\mathcal{P}}_n \cong \overline{\mathcal{B}}_{n+1}$.

We call a binary form of degree n of type $\{m_1, \ldots, m_r\}$ $(m_1 \geq m_2 \geq \cdots \geq m_r)$ if it has exactly r roots with multiplicities $m_1, m_2, \ldots, m_r$. According to [MFK94, p.78], a binary form of odd degree is semistable if and only if it is stable. Hence, it is enough to consider binary forms of even degrees.

Proposition 4.1. Suppose $n = 2m$ is even. Then there is exactly one closed orbit of strictly semistable binary forms of degree n in $\mathbb{P}^n$, namely the orbit of type $\{m, m\}$. In particular, $\overline{\mathcal{B}}_n$ is a one-point compactification of $\mathcal{B}_n$.

Proof. By Example 2.23, a binary form f is strictly semistable if and only if all roots have multiplicity m. We now show that all other orbits of semistable points have closure that degenerate to the orbit of binary forms of type $\{m, m\}$. This shows the uniqueness of the latter orbit. The actions of SL_2 allow us to move any root to infinity. So let us assume, without loss of generality, that the root of multiplicity m is at ∞. Hence f is of the form

$$f_n(x, y) = y^m(a_0x^m + a_1x^{m-1}y + \cdots + a_my^m), \quad a_0 \neq 0.$$

Then we have

$$\lambda(t)f(x, y) = t^{-m}y^m(a_0t^mx^m + a_1t^{m-1}x^{m-1}t^{-1}y) + \cdots + a_m(t^{-m}y^m),$$

so we get

$$\lim_{t\to\infty} \lambda(t)f_n(x, y) = a_0x^my^m.$$

Hence, the orbit of $a_0x^my^m$ is in the closure of the orbit of f_n. This means all strictly semistable binary forms degenerate to this point. The orbit of $a_0x^my^m$ is closed, hence it is the unique closed orbit corresponding to the strictly semistable binary form.

□

For the proof that $\overline{\mathcal{P}}_n \cong \overline{\mathcal{B}}_{n+1}$, we need the following lemma.

Lemma 4.2. Let $(A, B) \in W_n$ be a pair of matrices such that $P = \lambda A + \mu B$ is a pencil of quadrics of type $[(e_1, \ldots, e_{r_1}), (e_{r_1+1}, \ldots, e_{r_2}), \ldots, (e_{r_{s-1}+1}, \ldots, e_{r_s})]$. Suppose one of the numbers $e_i \geq 2$. Then the orbit $O((A, B))$ under the action of $SL_{n+1} \times SL_2$ is not closed.

Proof. It suffices to show that the orbit $O((A, B))$ is not closed under the action of SL_{n+1}. Without loss of generality, we may assume $e_1 \geq 2$. Let $(\lambda - a_1)^{e_1}, (\lambda - a_2)^{e_2}, \ldots, (\lambda - a_r)^{e_r}$ denote the elementary divisors of the pencil $\mathcal{P}$. According to the theory of normal forms, we may assume that A and B are of the form $A = diag(P_1, \ldots, P_r)$ and $B = diag(Q_1, \ldots, Q_r)$ where A and B are diagonal block matrices with blocks

$$P_i = \begin{pmatrix} & & & & a_i \\ & & & a_i & 1 \\ & & \iddots & \iddots & \\ & a_i & 1 & & \\ a_i & 1 & & & \end{pmatrix} \quad \text{and } Q_i = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

of size e_i. We assume $e_1 = 2l_1$ is even, the proof of the odd case being similar. The associated quadratic forms are

$$q_A := x^TAx = 2a_1X_1X_{e_1} + \cdots + 2a_1X_{l_i}X_{l_{i+1}} + 2X_2X_{e_1} + \cdots + 2X_{l_i}X_{l_{i+2}} + X_{l_{i+1}}^2 + f_A,$$
$$q_B := x^TBx = 2X_1X_{e_1} + \cdots 2X_{l_i}X_{l_{i+1}} + f_B$$

with quadratic forms $f_A = f_A(X_{e_1+1}, \ldots, X_{n+1})$ and $f_B = f_B(X_{e_1+1}, \ldots, X_{n+1})$. Consider the 1-parameter subgroup $\lambda : \mathbb{C}^* \to SL_{n+1}$ defined by

$$\begin{aligned}
\lambda(t)(X_i) &:= t^{-1}X_i \quad \text{for } 1 \le i \le l_1 \\
\lambda(t)(X_i) &:= tX_i \quad \text{for } l_1 + 1 \le i \le e_1 \\
\lambda(t)(X_i) &:= X_i \quad \text{for } e_1 + 1 \le i \le n + 1.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\lambda(t)q_A = 2a_1X_1X_{e_1} + \cdots + 2a_1X_{l_i}X_{l_{i+1}} + 2X_2X_{e_1} + \cdots + 2X_{l_i}X_{l_{i+2}} + t^2X_{l_{i+1}}^2 + f_A, \\
&\lambda(t)q_B = q_B.
\end{aligned}$$

Hence if we put

$$A_0 := \lim_{t\to 0} q_A = 2a_1X_1X_{e_1} + \cdots + 2a_1X_{l_i}X_{l_{i+1}} + 2X_2X_{e_1} + \cdots + 2X_{l_i}X_{l_{i+2}} + f_A,$$

we get that $(A_0, B) = \lim_{t\to 0}(A, B)$ is a pencil with one more elementary divisor than (A, B). □

Theorem 4.3. 1. The discriminant map $D : W_n \to \mathbb{P}_{n+1}$ induces an isomorphism

$$\overline{D} : \overline{\mathcal{P}}_n \to \overline{\mathcal{B}}_{n+1}.$$

2. The map $\overline{D}$ maps the open subset $\mathcal{P}_n \subset \overline{\mathcal{P}}_n$ into $\mathcal{B}_{n+1}$. The complement of its image in $\overline{\mathcal{B}}_{n+1}$, i.e., $\overline{\mathcal{B}}_{n+1} \setminus \overline{D}(\mathcal{P}_n)$ is the divisor $d = 0$ where d is the discriminant of binary forms of degree $n + 1$.

Remark. [AL00, Theorem 4.2] stated that the complement of $\mathcal{P}_n$ in $\mathcal{B}_{n+1}$ is the divisor $d = 0$. However it should be $\overline{\mathcal{B}}_{n+1}$, since $\overline{D}$ maps $\mathcal{P}_n$ isomorphically to $\mathcal{B}_{n+1}$.

Proof. (1): According to Lemma 3.6, the map $D : W_n \to \mathbb{P}^{n+1}$ is equivariant with respect to the actions, so it preserves orbits. Moreover, by Theorem 3.5, a point $(A, B) \in W_n$ is semistable if and only if $D(A, B)$ is semistable. Hence D induces a morphism $\overline{D}$ such that the following diagram commutes:

$$\begin{array}{ccc}
(W_n)^{ss} & \xrightarrow{\quad D \quad} & (\mathbb{P}^{n+1})^{ss} \\
\Big\downarrow{\scriptstyle \pi_p} & & \Big\downarrow{\scriptstyle \pi_b} \\
\overline{\mathcal{P}}_n & \xrightarrow[\quad \overline{D} \quad]{} & \overline{\mathcal{B}}_{n+1}
\end{array}$$

where

$$\pi_p : (W_n)^{ss} \to \overline{\mathcal{P}}_n := W_n//(SL_{n+1} \times SL_2)$$

and

$$\pi_b : (\mathbb{P}^{n+1})^{ss} \to \overline{\mathcal{B}}_{n+1} := \mathbb{P}^{n+1}//SL_2$$

are the natural maps associated to the GIT quotients. Recall that Zariski's Main Theorem states: a birational morphism of normal varieties which is bijective at points is an

isomorphism. The ring of invariants I_n of binary forms of degree n is known to be normal (see [Gey74, p.65]). Since $\overline{\mathcal{B}}_{n+1} = \mathrm{Proj}(I_{n+1})$, by Zariski's Main Theorem, it suffices to show that $\overline{D} : \overline{\mathcal{P}}_n \to \overline{\mathcal{B}}_{n+1}$ is bijective. Lemma 3.6 shows surjectivity, so we only need to show it is injective.

Let f be a binary form of degree $n+1$ of type $\{m_1, \ldots, m_r\}$, $m_1 \geq \cdots \geq m_r$. According to the theory of normal forms of pencils of quadrics in Section 3.1, Segre symbols classifies pencils of quadrics, i.e., for every Segre symbol

$$[(e_1, \ldots, e_{r_1}), (e_{r_1+1}, \ldots, e_{r_2}), \ldots, (e_{r_{s-1}+1}, \ldots, e_{r_s})]$$

with $\sum_{i=1}^{r_1} e_i = m_1, \ldots, \sum_{i=r_{s-1}+1}^{r_s} e_i = m_r$, there is exactly one orbit with this Segre symbol mapping to the orbit of f.

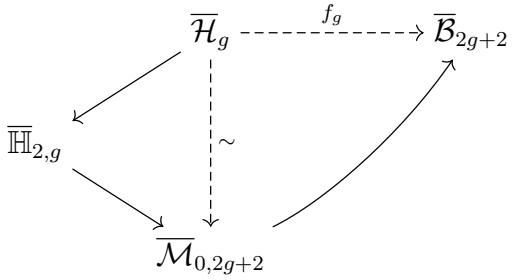
First, suppose that f is a stable binary form. We want to show that $\overline{D}^{-1}(\pi_b(f))$ contains only one point. Out of the orbits in $D^{-1}(O(f))$, Lemma 4.2 implies that there is exactly one closed orbit, namely the one of type $[(1, 1 \ldots, 1), (1, \ldots, 1), \ldots, (1 \ldots, 1)]$, where the i-th parentheses consist of m_i entries 1. Since points of $\overline{\mathcal{P}}_n$ parametrize the closed orbits of W_n (see Proposition 2.9), we get our claim.

Now let f be a strictly semistable binary form. Note that when $n+1$ is odd, there are no strictly semistable binary forms of degree $n+1$, so let us consider the case when $n+1 = 2m$ is even, then by Proposition 4.1, there is exactly one strictly semistable point p^{ss} in $\overline{\mathcal{B}}_{2m}$. whose orbit is of type $\{m, m\}$. Hence it remains to show that there is only one point in $\overline{\mathcal{P}}_{2m}$ mapping to p^{ss}. Since the orbit of $O(p^{ss})$ is of type $\{m, m\}$, the closed orbits in its preimage $D^{-1}(O(p^{ss}))$ must have Segre symbol with two pairs of parentheses $[(\cdots), (\cdots)]$, each of which correspond to a root, and the sum of the entries in each pair sum to m. By Lemma 4.2, the only option is $[(1, \ldots, 1), (1, \ldots 1)]$. This completes the proof.

(2): Recall that the discriminant of a binary form is zero if and only if it admits multiple roots. Then the statement follows from the fact that a pencil is strictly semistable (see Theorem 3.5) if and only if its discriminant binary form admits a multiple root. □

Remark. In the argument for when f is a strictly semistable binary form, we provide an alternative simplified proof by directly appealing to Lemma 4.2.

5 Binary Forms & Hyperelliptic Curves



In this section we show the relation between the moduli space of stable hyperelliptic curves of genus g $\overline{\mathcal{H}}_g$ and the moduli space of semistable binary forms of degree $2g+2$ $\overline{\mathcal{B}}_{2g+2}$:

Theorem 5.1. The canonical isomorphism $\mathcal{H}_g \xrightarrow{\sim} \mathcal{B}_{2g+2}$ extends to a holomorphic map $f_g : \overline{\mathcal{H}}_g \to \overline{\mathcal{B}}_{2g+2}$. For $g = 2$, f_2 is an isomorphism.

To do so, instead of constructing the map $\overline{\mathcal{H}}_g \to \overline{\mathcal{B}}_{2g+2}$, we use the isomorphism with the moduli space of stable n-marked curves of genus 0 $\overline{\mathcal{H}}_g \cong \overline{\mathcal{M}}_{0,2g+2}$ to construct a map $\overline{\mathcal{M}}_{0,2g+2} \to \overline{\mathcal{B}}_{2g+2}$.

5.1 $\overline{\mathcal{H}}_g$ and $\overline{\mathcal{M}}_{0,2g+2}$

Recall that a *stable (respectively semistable) n-pointed curve* is a complete connected curve B that has only nodes as singularities, together with an **ordered** collection $p_1, \ldots, p_n \in B$ of distinct smooth points such that every smooth rational component of the normalization of B has at least 3 (respectively 2) points lying over singular points or points among $p_1, \ldots, p_n$. Note that an n-pointed curve of genus 0 is stable if and only if it admits no nontrivial automorphisms. Similarly we define a *stable (respectively semistable) n-marked curve* to be a complete connected curve B that has only nodes as singularities, together with an **unordered** collection $p_1, \ldots, p_n \in B$ of distinct smooth points such that every smooth rational component of the normalization of B has at least 3 (respectively 2) points lying over singular points or points among $p_1, \ldots, p_n$. Note that any stable n-marked curve may admit nontrivial automorphisms, but its automorphism group is always finite. In the genus 0 case it is a subgroup of the symmetric group of order n.

Proposition 5.2. For $n \geq 3$, the coarse moduli space $\overline{\mathcal{M}}_{0,n}$ of stable n-marked curves of genus 0 exists.

Proof. According to [Knu83], the coarse moduli space $\overline{\mathbf{M}}_{0,n}$ of stable n-pointed curve of genus 0 exists (it is even a fine moduli space). The symmetric group S_n acts on $\overline{\mathbf{M}}_{0,n}$ by permuting the marked points. The group S_n is finite, so it preserves the coarse moduli structure, hence $\overline{\mathcal{M}}_{0,n} = \overline{\mathbf{M}}_{0,n}/S_n$ is a coarse moduli space of stable n-marked curves of genus 0. □

Definition 5.3. Let $(B, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n}$ be a stable n-marked curve of genus 0 and let $q_1, \ldots, q_k$ be the nodes of the curve B. An *admissible d-fold cover* of the curve $(B, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n}$ is a connected nodal curve C together with a regular map $\pi : C \to B$ such that:

1. The nonsingular points agree: $\pi^{-1}(B_{ns}) = C_{ns}$ and $\pi_{\pi^{-1}(B_{ns})}$ is a d-fold cover simply branched over the points p_i and otherwise unramified; and
2. The singular points agree: $\pi^{-1}(B_{sing}) = C_{sing}$ and for every node q of B and every node r of C lying over it, the two branches of C near r map to the branches of B near q with the same ramification index.

The definition of a family of admissible covers of a family of n-marked curves of genus 0 is analogous.

Proposition 5.4. For any $g \geq 1$ the coarse moduli space $\overline{\mathbf{H}}_{d,g}$ of admissable d-fold covers of stable $(2(g+d)-2)$-marked curves of genus 0, exists.

Proof. In [HM82] it is shown that admissable covers of a stable n-pointed curve of genus 0 was defined and that the coarse moduli space $\overline{\mathcal{H}}_{d,g}$ of admissable d-fold covers of stable $(2(g+d)-2)$-pointed curves of genus 0 exists. The symmetric group $S_{2(g+d)-2}$ acts on $\overline{\mathcal{H}}_{d,g}$ via permutation and since it is finite, the quotient $\overline{\mathbf{H}}_{d,g} = \overline{\mathcal{H}}_{d,g}/S_{2(d+g)-2}$ exists as a coarse moduli space. □

Lemma 5.5. Let $\pi : C \to B$ be an admissible double cover of a curve $(B, p_1, \ldots, p_{2n}) \in \overline{\mathcal{M}}_{0,2n}$.

(a) Every component of C is smooth.
(b) C is a semistable curve.
(c) The stable model C_s of C is obtained by contracting those rational components of C which intersect the other components of C in only 2 points.
(d) If C_i is a rational component of C intersecting the other components only in x_1 and x_2, then $\pi(x_1) = \pi(x_2)$.

Proof. (a) Every component of B is smooth. Since π is a regular double cover, every component of C is also smooth. (b) Suppose C is not semistable, so C contains a rational component C_i intersecting the other components of C in only one point, say x. Then x is a branch point of the cover π and $B_i = \pi(C_i)$ contains only one of the marked points, namely the other ramification point (whose existence is guaranteed by Riemann-Hurwitz). This contradicts the stability of B. (c) and (d): Let C_i be a component of C violating the stability of C. Since C is semistable, C_i is a rational component intersecting the other components of C in only 2 points. Suppose, for contradiction, that $\pi(x_1) \neq \pi(x_2)$. Then $B_i := \pi(C_i)$ has two special points (ramification points), contradicting the stability of B. Hence $\pi(x_1) = \pi(x_2)$ and the stable model is obtained by contracting C_i. □

Definition 5.6. Let C be a stable curve. We say that a nodal curve C' is *stably equivalent* to C if C is obtained from C' by contracting to a point all smooth rational components of C' meeting the other components of C' in only one or two points (that is, containing only one or two nodes of C').

Theorem 5.7 ([HM98, Thm.3.160])**.** A stable curve C is hyperelliptic if and only if there exists a double admissible cover $C' \to B$ of a stable pointed curve of genus 0 whose domain C' is stably equivalent to C.

Proposition 5.8. There is a canonical isomorphism

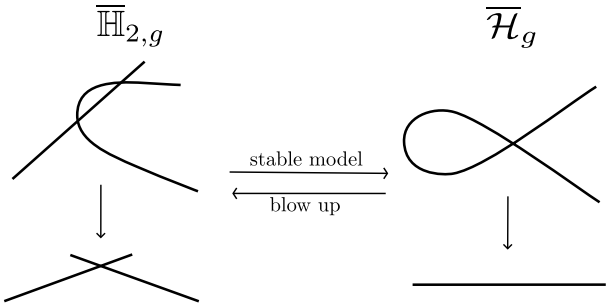
$$\varphi : \overline{\mathbf{H}}_{2,g} \to \overline{\mathcal{H}}_g$$

of the moduli space of admissible double covers of stable $(2g+2)$-marked curves and the moduli space of stable hyperelliptic curves of genus g.

Proof. Let $\pi : C \to B$ be an admissible double cover in $\overline{\mathbf{H}}_{2,g}$, then it is a limit of a family of smooth double covers of $\mathbb{P}_1$. C might not be stable, but it is a limit of smooth hyperelliptic curves, hence we define $\varphi : (\pi : C \to B)$ to be the stable model C_s of C. This map can be defined for families of admissible double covers, so we obtain a morphism $\overline{\mathbf{H}}_{2,g} \to \overline{\mathcal{H}}_g$.

To define an inverse morphism, let C be a stable hyperelliptic curve. According to Theorem 5.7, there is an admissible double cover $\pi : C' \to B$ of a stable marked curve B of genus 0 such that C' is stably equivalent to C. It follows from Lemma 5.5 that $\pi : C' \to B$ is uniquely determined by C. In fact, it is constructed as the blowup of the singular points of the components of C as well as all points where two curves of genus 0 meet. Hence we obtain a map $\psi : \overline{\mathcal{H}}_g \to \overline{\mathbf{H}}_{2,g}$. Since this definition extends to families of stable maps, ψ is a morphism. By construction, φ and ψ are inverses of each other. □

Remark. Given an element of $\overline{\mathcal{H}}_g$, there is a (semistable) admissible cover associated to it ([CH88]). As an example, a general element of Ξ_0 is illustrated on the right in the figure below. The curve is obtained by identifying two points that are conjugate under the hyperelliptic involution; it is not an admissible cover because the cover maps a node to a smooth point. By blowing up at the singular point, we obtain the admissible cover $\pi : C' \to B'$ on the left side. C' is composed of a rational component E and a smooth component of higher genus. E is mapped as a double cover to the corresponding rational component $\mathbb{P}^1$, branched not at the node but at two other points on $\mathbb{P}^1$. Note that E is strictly semistable because it only has two special points, so $C' \to B'$ is not in $\overline{\mathcal{H}}_g$.



Corollary 5.9. There is a canonical isomorphism

$$\Phi : \overline{\mathcal{H}}_g \xrightarrow{\sim} \overline{\mathcal{M}}_{0,2g+2}$$

of the moduli space $\overline{\mathcal{H}}_g$ of stable hyperelliptic curves genus g and the moduli space $\overline{\mathcal{M}}_{0,2g+2}$ of stable $(2g+2)$-marked curves of genus 0.

Proof. By Proposition 5.8 it suffies to show that there is a canonical isomorphism

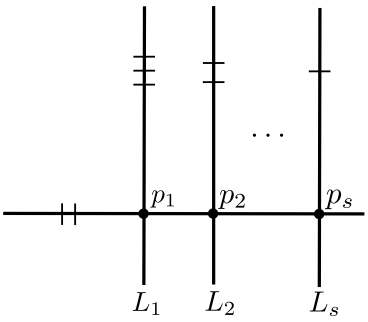
$$\varphi : \overline{\mathbf{H}}_{2,g} \to \overline{\mathcal{M}}_{0,2g+2}.$$

Let $\pi : C \to B$ be an element of $\overline{\mathbf{H}}_{2,g}$. Then by definition B is a stable $(2g+2)$-marked curve of arithmetic genus 0, giving a morphism $\varphi : \overline{\mathbf{H}}_{2,g} \to \overline{\mathcal{M}}_{0,2g+2}$.

Conversely, let us define a map $\psi : \overline{\mathcal{M}}_{0,2g+2} \to \overline{\mathbf{H}}_{2,g}$. Let $B \in \overline{\mathcal{M}}_{0,2g+2}$ be a stable $(2g+2)$-marked curve. We want to produce from B an admissible cover of a nodal curve in a unique way. Write $B = L_1 \cup \cdots \cup L_k$ where L_i are the irreducible components of B. Let $p_1, \ldots, p_{k-1}$ be the notes of B and $\alpha_1^j, \ldots, \alpha_{i_j}^j$ be the marks on L_j. Let $L_1, \ldots, L_s$ be the components of B that intersect the other components in only one point. Re-enumerating the L_i's if necessary, we may assume $p_1, \ldots, p_s$ are the only nodes of the components $L_1, \ldots, L_s$. We wish to assign a curve C_j to each L_j for $j = 1, \ldots, s$:

1. If i_j is even, then let C_j be the unique double cover of L_j branched over the points $\alpha_i^j, j = 1, \ldots, i_j$. The curve C_j will not be ramified over p_j.
2. If i_j is odd, then let C_j be the unique double cover of L_j branched over the points $\alpha_i^j, j = 1, \ldots, i_j$ and also over p_j.

This ensures that C_j always has even ramification points, which is required by the Riemann-Hurwitz formula. Now we consider the curve $\overline{B \setminus (L_1 \cup \cdots \cup L_s)}$. If it is nonempty, add one mark for each node that is a ramification point in the first step, and repeat the process above, determining in a unique way the double covers C_i over the components L_i ramified in the marked points of each component. The desired admissable



double cover $\psi(B)$ is given by $C = C_1 \cup \cdots \cup C_k / \sim$, glued at the nodes of B. One can see that ψ is a morphism, and by construction, φ and ψ are inverses of each other. $\square$

Since $\Phi(\mathcal{H}_g) = \mathcal{M}_{0,g+2}$, we can now investigate the boundary $\overline{\mathcal{H}}_g \setminus \mathcal{H}_g$ by pulling back the boundary $\overline{\mathcal{M}}_{0,2g+2} \setminus \mathcal{M}_{0,2g+2}$ via $\Phi : \overline{\mathcal{H}}_g \xrightarrow{\sim} \overline{\mathcal{M}}_{0,2g+2}$. According to [Kee92], the boundary $\overline{\mathcal{M}}_{0,2g+2} \setminus \mathcal{M}_{0,2g+2}$ is a divisor consisting of $g+1$ irreducible divisors D_i

$$\overline{\mathcal{M}}_{0,2g+2} \setminus \mathcal{M}_{0,2g+2} = D_1 \cup \cdots \cup D_{g+1},$$

where a general point of D_j represents a curve $B = B_1 \cup B_2$ with B_1 and B_2 isomorphic to $\mathbb{P}_1$ intersecting transversely in one point p and B_1 containing exactly j points of the marking. Define

$$\Delta_i := \Phi^{-1}(D_{2i+1}) \text{ for } i = 1, \ldots, \left[\frac{g}{2}\right]$$

and

$$\Xi_i := \Phi^{-1}(D_{2i+2}) \text{ for } i = 0, \ldots, [\frac{g-1}{2}].$$

Then by the definition of Φ, a general member of Δ_i is represented by an admissible

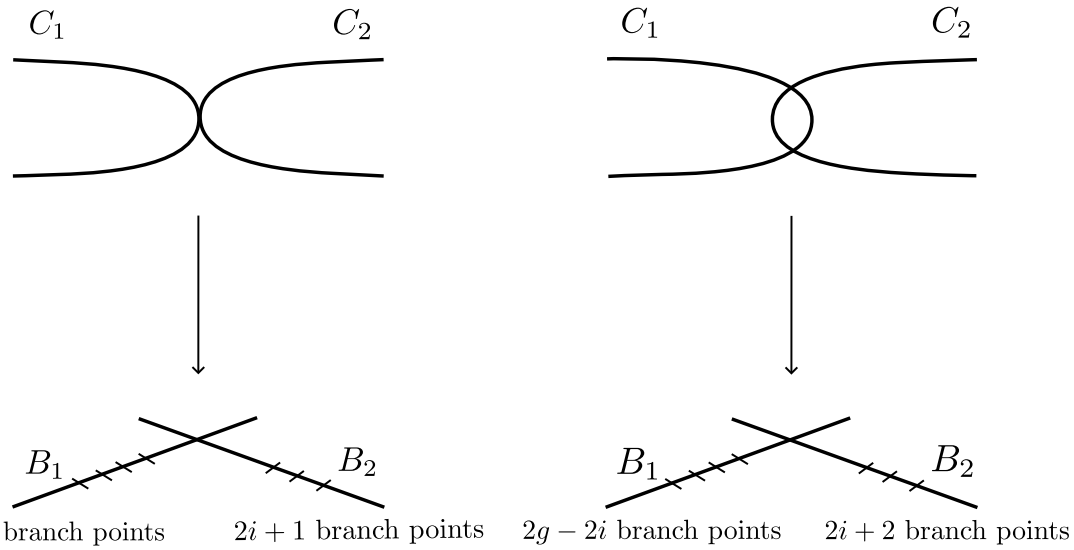


Figure 5: Boundary divisors Δ_i and Ξ_i of $\overline{\mathcal{H}}_g$.

cover on the left of Figure 5, and a general member of Ξ_i is represented by an admissible cover on the right of Figure 5. Note that the curve $C = C_1 \cup C_2$ is always stable except in

the case of Ξ_0 where C_1 has to be contracted to yield an irreducible rational curve with a node.

5.2 $\overline{\mathcal{M}}_{0,m}$ and $\mathcal{B}_m$

To show that there exists a holomorphic map $\overline{\mathcal{H}}_g \to \overline{\mathcal{B}}_{2g+2}$, by isomorphism in Corollary 5.9, it is equivalent to the following theorem:

Theorem 5.10. The map $\mathcal{M}_{0,m} \to \overline{\mathbf{B}}_m$ extends to a holomorphic map $F_m : \overline{\mathcal{M}}_{0,m} \to \overline{\mathbf{B}}_m$ for every $m \geq 3$.

The idea of the proof is to show that for any stable m-marked curve of genus 0 there is a unique "central component", to which one can contract all branches in order to obtain a stable binary form. For this, we need some preliminaries.

To an m-marked nodal curve $(C, p_1, \ldots, p_n)$ of genus 0 we can associate a weighted tree T in the following manner:

- Each vertex of the T corresponds to an irreducible component of C,
- two vertices are connected by an edge if the corresponding components intersect,
- the vertex v_i has weight $wt(v) := n_i$ where n_i denotes the number of marked points on the corresponding component of C.

Note that the resulting graph is acyclic and connected. The number m of marked points is called the *total weight* of T. This tree inherits the stability condition of a curve: a (connected) weighted tree is called *stable* if for every vertex c the sum of its weight and the number of edges starting at $v \geq 3$. Thus, a marked nodal curve of genus 0 is stable if and only if the corresponding tree T is stable.

Let v denote the vertex of an m-weighted tree T and $e_1(v), \ldots, e_n(v)$ the edges of T starting at v. The graph $T \setminus \{v, e_1(v), \ldots, e_n(v)\}$ consists of n weighted trees $T_1, \ldots, T_n$, which we call the subtrees *complementary* to the vertex v. A vertex v of an m-weighted tree T is called a *central vertex* if

$$wt(T_i) < m/2$$

for every subtree complementary to v.

Lemma 5.11. Let T denote a stable m-weighted tree with the property $(*)$:
There is no edge e of T such that for the 2 complementary subtrees T_1 and T_2 of e we have $wt(T_1) = wt(T_2) = \frac{m}{2}$.
Then T admits a unique central vertex v_{cent}.

Proof. To show the existence of v_{cent}, we use the following procedure: start at an arbitrary vertex v_1. If v_1 is not central, then there exists a unique subtree, say T_1^1, complementary to v_1 with $wt(T_1^1) > m/2$ (note that property $(*)$ implies $wt(T_1^1) \neq m/2$). Move to the adjacent vertex v_2 (the neighbor of v_1 in T_1^1). Let T_1^2 be the subtree complementary to v_2, which includes everything except T_1^1. Since $wt(T_1^1) > m/2$, the remaining weight is $wt(T_1^2) = m - wt(T_1^1) < m/2$. This ensures that at least one subtree of v_2 has weight $< m/2$.

Now we iterate the process: check the other subtrees of v_2. If any subtree T_j^2 has weight $\geq m/2$, move to the next vertex v_3 in that direction. The weight of the new complementary subtree T_1^3 will again satisfy $wt(T_1^3) < m/2$, because the total weight of

the previous "heavy" subtree is not split further. Since the tree is finite, the process must eventually reach a vertex v_{cent} where all complementary subtrees have weight $< m/2$.

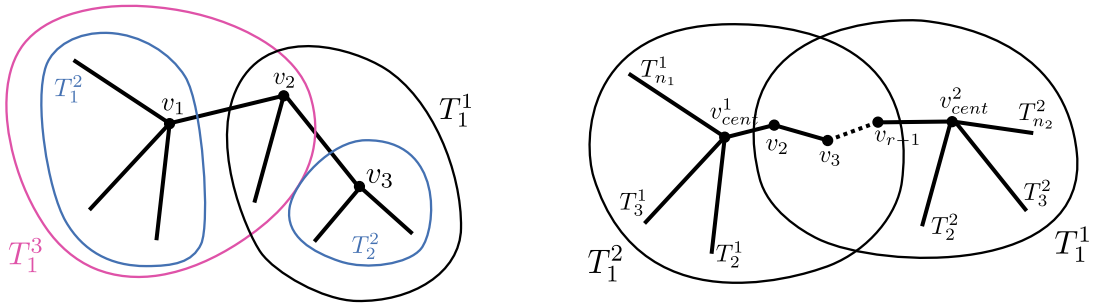


Figure 6: Illustrations for existence (left) and uniqueness (right).

Now we show the uniqueness of v_{cent}. Suppose we have two central vertices $v_{cent}^1 \neq v_{cent}^2$. Since T is a tree, there is a unique path $v_1 = v_{cent}^1, v_2, \ldots, v_r = v_{cent}^2$ between the two central vertices. Let $T_1^1, \ldots, T_{n_1}^1$ and $T_1^2, \ldots, T_{n_2}^2$ denote the subtrees complementary to v_{cent}^1 and v_{cent}^2. Without loss of generality, we may assume that T_1^1 is the subtree containing v_{cent}^2 and T_1^2 is the one containing v_{cent}^1. Then

$$\begin{aligned} m &> wt(T_1^1) + wt(T_1^2) \\ &\geq \sum_{i=1}^{r} wt(v_i) + \sum_{\nu=2}^{n_1} wt(T_\nu^1) + \sum_{\nu=2}^{n_1} wt(T_\nu^2) \\ &= wt(T) \\ &= m, \end{aligned}$$

where the first inequality is due to the vertices being central, hence all subtrees have weight $< m/2$. This is a contradiction.

□

A binary form of degree m can be considered as a smooth m-marked curve $(\mathbb{P}_1, x_1, \ldots, x_m)$ of genus 0. The markings $x_1, \ldots, x_m$ are given by the roots of f counted with multiplicities. The elements of the open dense set $\mathcal{M}_{0,m}$ may be considered as a binary form with no multiple roots. This induces a morphism $\mathcal{M}_{0,m} \to \overline{\mathbf{B}}_m$.

Proof of Theorem 5.10. Let $C \in F_m : \overline{\mathcal{M}}_{0,m}$ be a stable m-marked curve of genus 0 for some $m \geq 3$. We map the divisor $\Delta_{\frac{m}{2}}$ to the semistable point If $C \notin \Delta_{\frac{m}{2}}$ for even m, then we define the map as follows: Let T be the dual graph of C. T satisfies $(*)$ in Lemma 5.11, so it admits a central vertex v_{cent} dual to C_{cent}. This means that there are complementary trees $T_1, \ldots, T_r$ dual to $C_1, \ldots, C_r$. Let m_i denote the number of marked points of C_i. (Note that the condition means $m_i < m/2$.)

Now, choose coordinates $(x : y)$ of $C_{cent} \cong \mathbb{P}_1$. Contract $C_1, \ldots, C_r$ to the point $(x_i : y_i) := C_i \cap C$. Let $(x_{r+1} : y_{r+1}), \ldots, (x_s : y_s)$ be the marked points of C on C_{cent}. Then we define

$$F_m(C) := \prod_{i=1}^{r} (y_i X - x_i Y)^{m_i} \prod_{i=r+1}^{s} (y_i X - x_i Y).$$

Geometrically $F_m(C)$ correspond to the marked curve

$$(C_{cent}, (x_1 : y_1)^{m_1}, \ldots, (x_r : y_r)^{m_r}, (x_{r+1} : y_{r+1}), \ldots, (x_s : y_s)).$$

This process can be extended to holomorphic families: let $(\pi : C \to U, \sigma_1, \ldots, \sigma_m)$ denote a holomorphic family of stable m-marked curves of genus 0, then the central components of the fibers also form a holomorphic family $C_{cent} \to U$. Since the binary forms are determined from the central components, we also obtain a holomorphic family of binary forms $F_m(C \to U)$ over U. Thus we obtain a holomorphic map $F_m : \overline{\mathcal{M}}_{0,m} \to \overline{\mathcal{B}}_m$ for m odd and $F_m : \overline{\mathcal{M}}_{0,m} \setminus \Delta_{\frac{m}{2}} \to \overline{\mathcal{B}}_m$ for m even. But then certainly F_m extends to a continuous map on all of $\overline{\mathcal{M}}_{0,m}$ by just mapping the divisor $\Delta_{\frac{m}{2}}$ to the semistable point of $\overline{\mathcal{B}}_m$. Hence, by the Riemann Extension Theorem, F_m is holomorphic everywhere. □

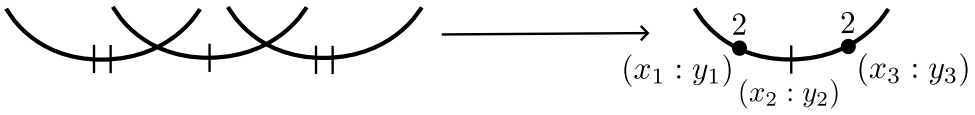


Figure 7: Under the map F_m, the curve contracts to its central component and remembers the markings with multiplicity.

Remark. It is important to note that F_m is not injective: Figure 8 shows two curves that both map to the same binary form. This is the main reason why $f_g : \overline{\mathcal{H}}_g \to \mathcal{B}_{2g+2}$ is not an isomorphism.

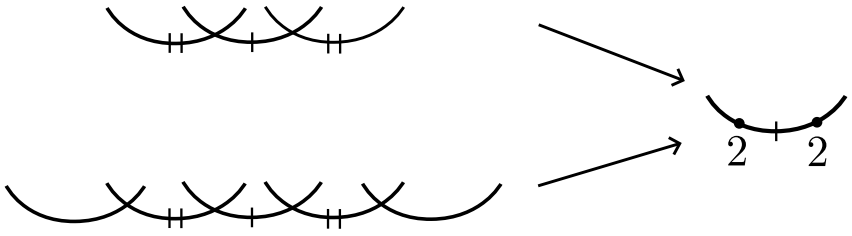


Figure 8: Non-injectivity of F_m.

Proposition 5.12. The holomorphic map f_g contracts Δ_i (respectively Ξ_i) to a subvariety of dimension $2g - 2i - 1$ (resp. $2g - 2i - 2$), except when $2i + 1 = g + 1$ (resp. $2i + 2 = g + 1$). In this case, f_g contracts Δ_i (resp. Ξ_i) to a point.

Proof. Let C be a general element of Δ_i and $C \to B$ the associated admissible cover. Then $B = B_1 \cup B_2$ where B_1 and B_2 are irreducible components of genus zero. If $2i+1 = g+1 = (2g+2)/2$, then f_g maps C to the semistable point. Otherwise, without loss of generality, we can assume that B_1 is the central component and therefore has $2g - 2i + 1$ marked points, say $x_1, \ldots, x_{2g-2i+1}$. Let x_0 denotes the point of intersection of B_1 and B_2. Since f_g is the composition $\overline{\mathcal{H}}_g \to \overline{\mathcal{M}}_{0,2g+2} \to \overline{\mathcal{B}}_{2g+2}$, according to Figure 9, C gets mapped to the binary form

$$f_g(C) = (X - x_0)^{2i+1}(X - x_1) \cdots (X - x_{2g-2i+1}).$$

Every binary form of this type is contained in the image $f_g(\Delta_i)$. Moreover, the forms of this type make up an open subset of the variety $f_g(\Delta_i)$. This implies

$$\dim(f_g(\Delta_i)) = 2g - 2i + 2 - \dim(\mathrm{Aut}(\mathbb{P}^1)) = 2g - 2i - 1,$$

since $\dim(\mathrm{Aut}(\mathbb{P}^1)) = PGL_1(\mathbb{C}) = 3$. The computation of $\dim(\Xi_i)$ is analogous. □

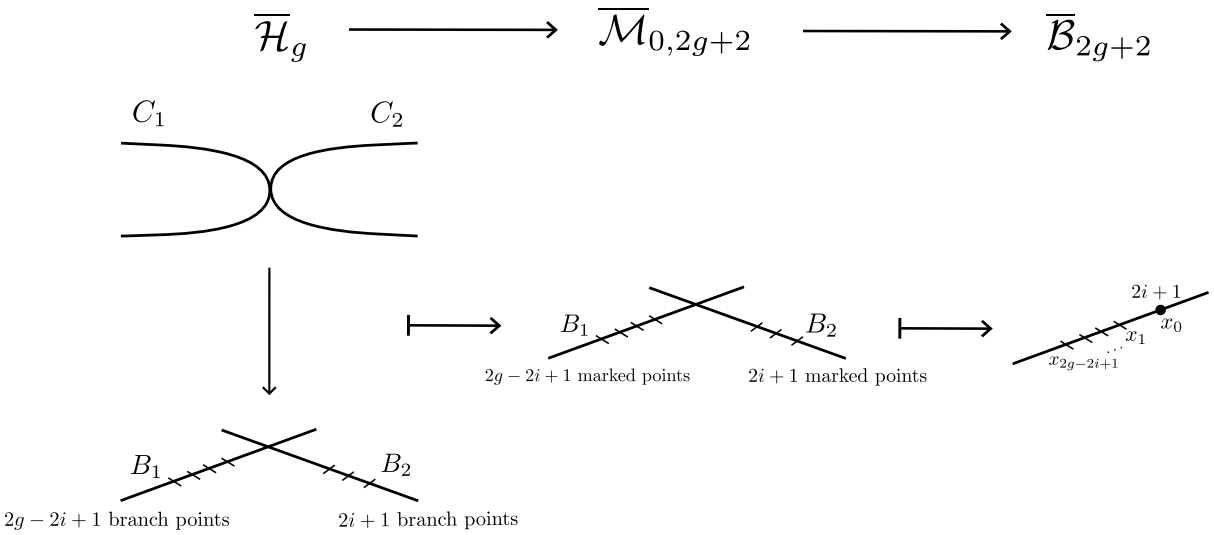


Figure 9: Dimension calculation of f_g.

Remark. [AL01, Section 4] determines $f_g(C)$ for $C \in \Delta_i$ by finding the stable models of the admissible covers. We omit the calculation by reading off the branch data directly from the base curves of the stable hyperelliptic curves.

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